Theoretical Computer Science 603 (2015) 58–83

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

The complexity of one-agent refinement modal logic

Laura Bozzelli [a], Hans van Ditmarsch [b,*], Sophie Pinchinat [c]

[a] Technical University of Madrid (UPM), Madrid, Spain
[b] LORIA, CNRS/University of Lorraine, France
[c] IRISA/INRIA, University of Rennes, France

ARTICLE INFO

Article history:
Received 14 February 2013
Accepted 7 May 2014
Available online 15 July 2015

Keywords:
Modal logic
Complexity of satisfiability
Bisimulation quantifiers

ABSTRACT

We investigate the complexity of satisfiability for one-agent *refinement modal logic* (RML), a known extension of basic modal logic (ML) obtained by adding refinement quantifiers on structures. It is known that RML has the same expressiveness as ML, but the translation of RML into ML is of non-elementary complexity, and RML is at least *doubly* exponentially more succinct than ML. In this paper, we show that RML-satisfiability is 'only' *singly* exponentially harder than ML-satisfiability, the latter being a well-known PSPACE-complete problem. More precisely, we establish that RML-satisfiability is complete for the complexity class $\mathsf{AEXP}_{\mathsf{pol}}$, i.e., the class of problems solvable by alternating Turing machines running in single exponential time but only with a polynomial number of alternations (note that $\mathsf{NEXPTIME} \subseteq \mathsf{AEXP}_{\mathsf{pol}} \subseteq \mathsf{EXPSPACE}$).[1]

1. Introduction

Modal logics with explicit or implicit propositional quantification Refinement modal logic is a logic with propositional quantification. Modal logics augmented with propositional quantifiers, which allow to quantify over subsets of the domain of the current model, have been investigated since Fine's seminal paper [2]. Fine distinguishes three different propositional quantifications, which allow different kinds of model transformations: quantifying over propositionally definable subsets (over booleans), quantifying over subsets definable in the logical language (of basic modalities and quantifiers), and quantifying over all subsets. Only the first two are, in our modern terms, bisimulation preserving. Propositional quantification can easily lead to undecidable logics [2,3]. Undecidability relies on the ability of propositional quantification to dictate the structural properties of the underlying model [3]. This has motivated, more recently, the introduction of bisimulation quantified logics [4,5,3,6]. In that framework, the quantification is over the models which are bisimilar to the current model except for a propositional variable p. This operation is bisimulation preserving, and these logics are decidable.

In [7] the authors propose a novel way of quantifying, namely over modally definable submodels. Unlike the above proposals, this not merely involves changing the valuation of a proposition in a subdomain, but *restricting the model* to that subdomain. The setting for these logics is how to quantify over information change. In the logic APAL of [7], an expression that we might write as $\exists\varphi$ for our purposes stands for 'there is a modal formula ψ such that in the submodel restriction to the states satisfying ψ it holds that φ'. This logic is undecidable [8].

* Corresponding author.
E-mail addresses: laura.bozzelli@fi.upm.es (L. Bozzelli), hans.van-ditmarsch@loria.fr (H. van Ditmarsch), Sophie.Pinchinat@irisa.fr (S. Pinchinat).

1 This work is the revised and expanded version of [1].

http://dx.doi.org/10.1016/j.tcs.2015.07.015

Refinement modal logic (RML) [9–11] is a generalization of this perspective to more complex model transformations than submodel restrictions. This is achieved by existential and universal quantifiers which range over the *refinements* of the current model. In RML, an expression $\exists_r\varphi$ stands for 'there is a refinement wherein it holds that φ.' Given a model and a refinement of that model, we say that the two are in the *refinement relation*. From the *atoms/forth/back* requirements of bisimulation, a refinement (relation) between two given modal structures needs only satisfy *atoms* and *back*. Refinement is therefore the dual of a simulation that needs only satisfy *atoms* and *forth*, and it is more general than model restriction, since it is equivalent to bisimulation followed by model restriction. From a syntactic point of view, a *refinement formula* of the form $\exists_r\varphi$ can be mimicked by an existential bisimulation quantification followed by a *relativization* of φ, that is a simple syntactic transformation of φ which holds whenever φ holds in the outcome of this bisimulation quantification [11]. Just as in bisimulation quantified logics we have *explicit* quantification over propositional variables, refinement quantification as it is realized in refinement modal logic is *implicit* quantification over propositional variables, i.e., quantification over variables not occurring in the formula bound by the quantifier.

As an example of a refinement consider the following four rooted (underlined) structures.

$$
\begin{array}{rl}
\underline{\bullet} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet & M \\
\underline{\bullet} \longrightarrow \bullet \longrightarrow \bullet & M' \\
\bullet \longleftarrow \underline{\bullet} \longrightarrow \bullet \longrightarrow \bullet & M'' \\
\bullet \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \underline{\bullet} \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet & M'''
\end{array}
$$

With respect to the first model, M, the second one, M', is a model restriction. Model M'' is a refinement of M. It is not a model restriction. However, it is a model restriction of M''', a bisimilar copy of M. Refinements have really different properties, e.g., a formula like $\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot$ is clearly false in any model restriction of M, but it is true in its refinement M''. The root of the original model M satisfies the formula $\exists_r(\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot)$, where $\exists_r$ is the refinement quantifier.

As amply illustrated in [11], refinement quantification has applications in many settings: in logics for games [12,6], it may correspond to a player discarding some moves; for program logics [13], it may correspond to operational refinement; and for logics for spatial reasoning, it may correspond to subspace projections [14].

Our contribution We now get to the content of this paper and its novel contributions. We focus on complexity issues for (one-agent) refinement modal logic [9–11], the extension of (one-agent) basic modal logic (ML) obtained by adding the existential and universal refinement quantifiers $\exists_r$ and $\forall_r$.[2] It is known [10,11] that RML has the same expressivity as ML, but the translation of RML into ML is of non-elementary complexity (see Section 6 in [10]) and no elementary upper bound is known for its satisfiability problem [11]. In fact, an upper bound in 2EXPTIME has been claimed in [10] by a tableaux-based procedure: the authors later concluded that the procedure is sound but not complete [11]. In this paper, our aim is to close that gap. We also investigate the complexity of satisfiability for some equi-expressive fragments of RML. In particular, we associate with each RML formula φ a parameter $\Upsilon_w(\varphi)$ corresponding to a slight variant of the classical quantifier alternation depth (measured w.r.t. $\exists_r$ and $\forall_r$), and for each $k \geq 1$, we consider the fragment RML^k consisting of the RML formulas φ such that $\Upsilon_w(\varphi) \leq k$. Moreover, we consider the existential (resp., universal) fragment $\mathsf{RML}^\exists$ (resp., $\mathsf{RML}^\forall$) obtained by disallowing the universal (resp., existential) refinement quantifier.

In order to present our results, first, we recall some computational complexity classes. We assume familiarity with the standard notions of complexity theory [15,16]. We will make use of the levels Σ_k^{EXP} ($k \geq 1$) of the exponential-time hierarchy EH, which are defined similarly to the levels Σ_k^{P} of the polynomial-time hierarchy PH, but with NP replaced with NEXPTIME. In particular, Σ_k^{EXP} corresponds to the class of problems decided by single exponential-time bounded Alternating Turing Machines (ATM, for short) with at most $k-1$ alternations and where the initial state is existential [15]. Note that $\Sigma_1^{\mathsf{EXP}} = \mathsf{NEXPTIME}$. Recall that $\mathsf{EH} \subseteq \mathsf{EXPSPACE}$ and EXPSPACE corresponds to the class of problems decided by single exponential-time bounded ATM (with no constraint on the number of alternations) [17]. We are also interested in an intermediate class between EH and EXPSPACE, here denoted by $\mathsf{AEXP}_{\mathsf{pol}}$, that captures the precise complexity of some relevant problems [18,15,19] such as the first-order theory of real addition with order [18,15]. Formally, $\mathsf{AEXP}_{\mathsf{pol}}$ is the class of problems solvable by single exponential-time bounded ATM with a polynomial-bounded number of alternations.[3]

Our complexity results are summarized in Fig. 1 where we also recall the well-known complexity of ML-satisfiability. For the upper bounds, the (technically non-trivial) main step in the proposed approach exploits a "small" size model property: we establish that like basic modal logic ML, RML enjoys a single exponential size model property. Note that our approach is completely different from the one proposed in [10]. There, a tableaux-based algorithm is given in a game-theoretic setting which is sound but not complete. The main reason of the incompleteness is that in the tableaux construction, the refinement quantification is just applied to model restrictions of the current "syntactical" model. In our approach instead, we use a tableaux construction only to infer a single exponential size model property. In particular, our tableaux construction

[2] Refinement modal logic is called "Future Event Logic" in [10].

[3] In Presburger arithmetic there are complexity classes of the form $\mathsf{STA}(f(n), g(n), h(n))$, where S is for Space, T for Time and A for Alternation, and where f, g, and h are functions. In that notation, $\mathsf{AEXP}_{\mathsf{pol}}$ is the class for any f (denoted by $*$), a g exponential in n and an h polynomial in n. See [20].