Theoretical Computer Science 603 (2015) 84–110

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Bivalent semantics, generalized compositionality and analytic classic-like tableaux for finite-valued logics ☆

Carlos Caleiro [a,*], João Marcos [b,*], Marco Volpe [c,*]

[a] SQIG, Instituto de Telecomunicações and Dept. Mathematics, IST, U Lisboa, Portugal
[b] LoLITA and Dept. of Informatics and Applied Mathematics, UFRN, Brazil
[c] Dipartimento di Informatica, Università di Verona, Italy

ARTICLE INFO

Article history:
Accepted 22 May 2014
Available online 17 July 2015

Keywords:
Bivalent semantics
Truth-functionality
Compositionality
Analyticity
Tableaux
Proof complexity

ABSTRACT

The paper is a contribution both to the theoretical foundations and to the actual construction of efficient automatizable proof procedures for non-classical logics. We focus here on the case of finite-valued logics, and exhibit: (i) a mechanism for producing a classic-like description of them in terms of an effective variety of bivalent semantics; (ii) a mechanism for extracting, from the bivalent semantics so obtained, uniform (classically-labeled) cut-free standard analytic tableaux with possibly branching invertible rules and paired with proof strategies designed to guarantee termination of the associated proof procedure; (iii) a mechanism to also provide, for the same logics, uniform cut-based tableau systems with linear rules. The latter tableau systems are shown to be adequate even when restricted to analytic cuts, and they are also shown to polynomially simulate truth-tables, a feature that is not enjoyed by the former standard type of tableau systems (not even in the 2-valued case). The results are based on useful generalizations of the notions of analyticity and compositionality, and illustrate a theory that applies to many other classes of non-classical logics.

1. Introduction

Our paper is a contribution to the modern study of deduction in many-valued logics, in line with the research from standard references such as [3,21], and consummating the track of publications surveyed in [9]. The present paper deals with finite-valued logics — logics whose connectives are semantically characterizable by truth-tables with a finite number of 'algebraic truth-values'. We first recall that such logics may be alternatively characterized by way of bivalent semantics — semantics with only two 'logical values' (cf. [32,10]). Going beyond that, we show that such bivalent characterizations, based on a generalized notion of compositionality, can be produced in a constructive way, for any finite-valued logic. Several technical problems that appear underway are shown to be circumventable. Providing further evidence on how model-theoretic and proof-theoretic analyses have strong impact on each other, from our bivalent characterizations of finite-valued logics we show, in each case, how to extract adequate analytic classic-like tableau systems. Analyticity, in these systems, is based on

☆ The research reported in this paper falls within the scope of the EU FP7 Marie Curie PIRSES-GA-2012-318986 project GeTFun: *Generalizing Truth-Functionality*. The first author further acknowledges the support of FCT and EU FEDER via the project PEst-OE/EEI/LA0008/2013 of Instituto de Telecomunicações. The second author acknowledges partial support of CNPq via the project GeVe (482809/2013-2).

* Corresponding authors.
E-mail addresses: ccal@math.ist.utl.pt (C. Caleiro), jmarcos@dimap.ufrn.br (J. Marcos), marco.volpe@univr.it (M. Volpe).

http://dx.doi.org/10.1016/j.tcs.2015.07.016
0304-3975/

appropriate generalized versions of the subformula property and on the adoption, in each case, of convenient proof strategies. While analytic tableaux for propositional logic are expected to yield decidability, there is no general reason to expect the associated decision procedure to be computationally feasible. In order to secure a measurable gain in proof complexity we show also how to extract, from our bivalent characterizations of finite-valued logics, alternative tableau systems that control the combinatorial explosion caused by intrinsic redundancies of usual analytic tableau methods. We show that these alternative systems can polynomially simulate truth-tables, the former thus not being 'worse' than the latter. Such cut-based tableaux generalize the so-called 'KE system' for Classical Logic (cf. [14]), in which all tableau rules are linear except for the (non-eliminable yet analytic) cut rule.

In Section 2 we list the basic syntactical definitions about logics in general and the basic semantic definitions about finite-valued logics in particular, and contrast truth-functional semantics with classic-like (bivalent) semantics. Several well-known examples of truth-functional logics are introduced. Many-valued logics in general, and truth-functional logics in particular, are shown to be (non-constructively) reducible to bivalent semantics, alongside the lines of the so-called 'Suszko's Thesis'. To render such bivalent reduction constructive, for a given finite-valued truth-functional logic, a fine analysis of its expressiveness is due: in turn, we show how one may algorithmically check for sufficient expressiveness, and how one may generate upon demand a sufficiently expressive conservative extension of the given logic. We then show how to produce an adequate classic-like characterization of any given finite-valued logic. We also show that this characterization is rather robust: the collection of boolean statements that determines it, in the metalanguage, may be replaced by equivalent (and possibly more economical) collections of similar statements. Our reductive mechanism gives rise to an effective variety of bivalent semantics, based on a generalization of the syntactical notion of subformula and a related broader take on the Principle of Compositionality of Meaning.

In Section 3 we show how to deal with partial information and syntactical subtleties describing unobtainable semantic scenarios, that will lead to nonstandard additional closure rules in tableau systems. Adequate classic-like tableaux are then shown to be extractible for each sufficiently expressive finite-valued logic. An extended notion of analyticity is guaranteed by a proof strategy to be coupled with a given proof system, based on an extended notion of formula complexity.

On the one hand, it is well known that proofs involving the cut rule (or, equivalently, modus ponens) can be dramatically shorter than the shortest cut-free proof of the same assertion (see, e.g., [6], and the discussion in [14, Section 3.8], where the introduction of cut-based KE tableau systems is motivated). On the other hand it is obvious that unrestricted use of cuts may lead to infinitary branching in proof search. Taking those facts into account, restricted forms of cut have been investigated that imply gains in minimal proof size without rendering proof search unwieldy. In particular, cut-based tableaux are based on a goal-directed form of employing analytic cuts, that is, cuts involving what we call generalized subformulas of formulas already to be found in a given branch. Cut-based tableaux for Classical Logic are studied in [15]. In Section 4 of the present paper we show how such systems may be uplifted to the realm of finite-valued logics. Moreover, for Classical Logic it has been proved (cf. [13]) that (propositional) cut-based tableaux polynomially simulate the truth-table procedure while for some classes of formulas the shortest standard analytic tableaux may be exponentially larger than the truth-tables. We extend these findings about proof complexity to finite-valued logics in general.

2. Exploring the bivalence behind truth-functionality

In what follows we propose a mechanism for producing a classic-like description of an arbitrary finite-valued logic in terms of an effective variety of bivalent semantics. To accomplish such goal, we show how one may exploit the linguistic resources of a given logic, automatically checking for its sufficient expressiveness, and minimally extending it, in a conservative way, when necessary.

2.1. Finite-valued logics

Consider an alphabet consisting of a denumerable set $\mathcal{A}$ of *atomic variables* and a finite collection Σ of *connectives* (or *constructors*). By $\Sigma_k \subseteq \Sigma$ we will denote the collection of k-ary constructors in Σ; the 0-ary connectives are also called *sentential constants*. The set $\mathcal{S}$ of *formulas*, as usual, is the carrier of the free Σ-algebra $\mathbb{S}$ generated by $\mathcal{A}$. By $\varphi(q_1, \ldots, q_k)$ we will denote a *statement-form* $\varphi \in \mathcal{S}$ written in the variables $q_1, \ldots, q_k \in \mathcal{A}$; if $\psi = \varphi(\psi_1, \ldots, \psi_k)$, for given $\psi_1, \ldots, \psi_k \in \mathcal{S}$, we say that ψ is an *instance of* φ. By $\mathcal{S}(\varphi)$ we denote the set of all instances of φ. If $\varphi \in \mathcal{S}$ contains some k-ary connective, for $k > 1$, we call this formula *composite*; otherwise, that is, in case φ is either an atomic variable or a sentential constant, we call it *noncomposite*. The outermost constructor of a composite formula is called its *head* connective. Formulas containing no atomic variables are called *ground*. Given $\varphi = \odot(\varphi_1, \varphi_2, \ldots, \varphi_k)$ in $\mathcal{S}$, with $\odot \in \Sigma_k$, we call $\varphi_1, \ldots, \varphi_k \in \mathcal{S}$ the *immediate subformulas* of φ. The set $\mathsf{sbf}(\varphi)$ of *subformulas of* φ is obtained by closing $\{\varphi\}$ under immediate subformulas, that is, it is the smallest set containing φ and the immediate subformulas of each element of $\mathsf{sbf}(\varphi)$. A *proper subformula of* φ is any element of $\mathsf{sbf}(\varphi) \setminus \{\varphi\}$. These notions are extended from formulas to sets of formulas in the usual way. A canonical way of measuring the complexity of a given formula is by counting the nested occurrences of k-ary constructors in it, for $k > 1$, that is, by inductively defining a mapping $\mathsf{dpth} : \mathcal{S} \longrightarrow \mathbb{N}$ such that:

$$\mathsf{dpth}(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is noncomposite} \\ 1 + \underset{1 \le i \le k}{\mathsf{Max}}\ \mathsf{dpth}(\varphi_i) & \text{if } \varphi = \odot(\varphi_1, \ldots, \varphi_k) \\ & \quad \text{for } \odot \in \Sigma_k, k \neq 0 \text{ and } \varphi_1, \ldots, \varphi_k \in \mathcal{S} \end{cases}$$