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Simple dynamics on graphs

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ABSTRACT

Can the interaction graph of a finite dynamical system force this system to have a "complex" dynamics? In other words, given a finite interval of integers A, which are the signed digraphs G such that every finite dynamical system $f : A^n \to A^n$ with G as interaction graph has a "complex" dynamics? If $|A| \geq 3$ we prove that no such signed digraph exists. More precisely, we prove that for every signed digraph G there exists a system $f : A^n \to A^n$ with G as interaction graph that converges toward a unique fixed point in at most $\lfloor \log_2 n \rfloor + 2$ steps. The boolean case $|A| = 2$ is more difficult, and we provide partial answers instead. We exhibit large classes of unsigned digraphs which admit boolean dynamical systems which converge toward a unique fixed point in polynomial, linear or constant time.

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1. Introduction

Let $A = \{0, 1, \dots, s\}$ be a finite integer interval, and let n be a positive integer. A *finite dynamical system* is a function

$$f : A^n \to A^n, \qquad x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x)).$$

If $|A| = 2$, such a system is called *boolean network*. Finite dynamical systems, and boolean networks in particular, have many applications: they have been used to model gene networks [16,25,26,15], neural networks [17,14,7,8], social interactions [19,12] and more (see [27,10]).

The structure of a finite dynamical system f can be represented via its *interaction graph* G, which roughly describes the dependencies between the variables of the systems (depending on the context, this graph is sometimes called *dependency graph*, *influence graph* or *regulatory graph*). More formally, G is a digraph with vertex set $\{1, \dots, n\}$ and an arc from j to i if $f_i(x)$ depends on x_j. An arc from j to i can also be labeled by a sign indicating whether $f_i(x)$ is an increasing (positive sign), decreasing (negative sign), or non-monotone (zero sign) function of x_j. This is commonly the case when modeling gene networks, since a gene can typically either activate (positive sign) or inhibit (negative sign) another gene.

In many contexts, as in molecular biology, the interaction graph is known—or at least well approximated—, while the actual function f is not. A natural and difficult question is then the following: *what can be said on system $f : A^n \to A^n$ according to its interaction graph only?* Among the many dynamical properties that can be studied, fixed points are crucial

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because they represent stable states [21,27,8]. As such, they are arguably the property which has been the most thoroughly studied (see [21,23,20,1,11,5] and the references therein).

In this paper, we are interested in "simple" dynamics, considering that a dynamics is simple if it describes a fast convergence toward a unique fixed point. Formally, f converges towards a unique fixed point in k steps if f^k is a constant. In that case, we say that f is a *nilpotent function* and the minimal k such that f^k is a constant is called the *class* of f. Also, we say that a signed or unsigned digraph G *admits* a function f if G is the signed or unsigned version of the interaction graph of f.

A fundamental result of Robert is the following: *if the interaction graph of $f : A^n \to A^n$ is acyclic then f is a nilpotent function of class at most n* [21]. This shows that "simple" interaction graphs imply "simple" dynamics. But conversely, does "complex" interaction graphs imply "complex" dynamics? More precisely, which are the interaction graphs that can force a system to have a non-simple dynamics? This is the question we study in this paper.

We first study the non-boolean case $|A| \geq 3$ in Section 3. Essentially, we show that every signed digraph G on n vertices admits a nilpotent function $f : A^n \to A^n$ of class at most $\lfloor \log_2 n \rfloor + 2$. Furthermore, if $|A| > 3$ then the upper-bound on the class of f can be reduced to only 2. Hence, in the non-boolean case, we cannot conclude that a system f has a non-simple dynamics from its interaction graph only.

We then study the boolean case $|A| = 2$ in Section 4, which is more difficult. First, not all digraphs admit a boolean nilpotent function. The directed cycle is the most simple example, and it seems very difficult to characterize the digraphs that admit a boolean nilpotent function. Thus we provide partial answers. We exhibit large classes of unsigned digraphs which admit boolean dynamical systems which converge toward a unique fixed point in polynomial, linear or constant time. In particular, we prove that if G has a primitive spanning strict subgraph then G admits a boolean nilpotent function of class at most $n^2 - 2n + 3$. We also prove that if G is strongly connected and if the out-neighborhood of some vertex of G induces a non-acyclic digraph, then G admits a boolean nilpotent function f of class at most $2n - 1$. Besides, we prove that if G is a loop-less connected symmetric digraph with at least three vertices, then G admits a boolean nilpotent function f of class 3. We have not been able to prove or disprove the following assertion: there exists a constant c such that, for every digraph G with n vertices, if G admits a boolean nilpotent function, then G admits a boolean nilpotent function of class at most cn.

2. Preliminaries

The vertex set of a digraph G is denoted $V(G)$ and its arc set, which is a subset of $V(G) \times V(G)$, is denoted $A(G)$. The in-neighborhood of a vertex v is denoted $G(v)$; this is an non-usual but very convenient notation for our purpose. Other notations and terminologies on digraphs are usual and consistent with [2]. Paths and cycles of are always directed, without repetition of vertices, and seen as subgraphs. The subgraph of G induced by a set of vertices $I \subseteq V(G)$ is denoted $G[I]$. If X is an arc, a vertex, a set of arcs, or a set of vertices, then $G \setminus X$ is the subgraph obtain from G by removing X or the elements in X. We say that G is *strong* if G is strongly connected. A strongly connected component I (*strong component* for short) of G is *initial* if there is no arc (u, v) with $u \notin I$ and $v \in I$. If G and G' are two digraphs, then $G \cup G'$ is the digraph with vertex set $V(G) \cup V(G')$ and arc set $A(G) \cup A(G')$. A digraph *on* a set V is a digraph with vertex set V. A *tree* is a digraph in which all the vertices have in-degree one, excepted one vertex, called the *root*, which has in-degree zero. A *forest* is a digraph in which all the connected components are trees. A *loop* is an arc from a vertex to itself. A vertex is *linear* if it has a unique in-neighbor and a unique out-neighbor.

A signed digraph G consists in a digraph, denoted $|G|$, together with a map that labels each arc of $|G|$ by a positive, negative or null sign. We say that an arc is *signed* if it is positive or negative, and *unsigned* otherwise. The digraph obtained from G by keeping only positive arcs is denoted G^+. We define similarly G^- and G^0. The digraph obtained by keeping only signed arcs is denoted $G^{\pm}$ (thus $G^{\pm} = G^+ \cup G^-$). A cycle of G is positive (resp. negative) if it contains an unsigned arc or an even (resp. odd) number of negative arcs. In the following, all graph-theoretic concepts that do not involve signs are applied on G or $|G|$ indifferently.

Let A be a finite interval of integers, let n be a positive integer and $[n] = \{1, \dots, n\}$. A function *over* A is a map $f : A^n \to A^n$. A function over $\{0, 1\}$ is a *boolean function*. As usual, for all $k \in \mathbb{N}$ we set $f^k = \mathrm{id}$ if $k = 0$ and $f^k = f \circ f^{k-1}$ otherwise. If f is any function, we write $f = \mathrm{cst}$ to mean that f is a constant. In the following, functions are often defined using conjunctions ($\wedge$) disjunctions ($\vee$) and exclusive disjunctions ($\oplus$). If $I \subseteq [n]$ and $x \in \{0, 1\}^I$ then, by convention, $\vee_{i \in I} x_i = \oplus_{i \in I} x_i = 0$ and $\wedge_{i \in I} x_i = 1$ if I is empty, and $\vee_{i \in I} x_i = \oplus_{i \in I} x_i = \wedge_{i \in I} x_i = x_i$ if $I = \{i\}$.

Definition 1. A function f over A is *nilpotent* if there exists $k \in \mathbb{N}$ such that f^k is constant. If f is nilpotent, then the smallest k such that f^k is a constant is the *class* of f.

Definition 2. The *interaction graph* of a function f over A is the signed digraph $G(f)$ on $[n]$ with arcs defined as follows: for all $j, i \in [n]$, there is an arc (j, i) if $f_i(a) \neq f_i(b)$ for some $a, b \in A^n$ such that $a_j < b_j$ and $a_k = b_k$ for all $k \neq j$; and an arc (j, i) is positive if $f_i(a) \leq f_i(b)$ for all such a and b, negative if $f_i(a) \geq f_i(b)$ for all such a and b, and null otherwise.

Hence, $G(f)$ has an arc (j, i) if and only if $f_i(x)$ depends essentially on x_j, and the sign of an arc (i, j) is positive (resp. negative) if an only if for every fixed x_k, $k \neq j$, $f_i(x)$ is a non-decreasing (resp. non-increasing) function of x_j.

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