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Sparse convolution-based digital derivatives, fast estimation for noisy signals and approximation results ☆

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ABSTRACT

We provide a general notion of a Digital Derivative in 1-dimensional grids, which has real or integer-only versions. From any such masks, a family of masks called skipping masks are defined. We prove general results of multigrid convergence for skipping masks. We propose a few examples of digital derivative masks, including the now well-known binomial mask. The corresponding skipping masks automatically have multigrid convergence properties. We study the cases of parametric curves tangents and curvature. We propose a novel interpretation of digital convolutions as computing points on a smooth curve, the regularity of which depends on the mask. We establish, in the case of binomial and B-spline masks, a close relationship between the derivatives of the smooth curve, and the digital derivatives provided by the mask.

1. Introduction

1.1. Aim of the work

The goal of this work is to put in place a foundation stone for a new paradigm for numerical calculations. The motivation for this is that the objects of current methods of numerical analysis, which deal mathematically primarily with real numbers, cannot effectively be computed, or even stored in memory with state of the art computers, even taking account the possibilities of more powerful machines implementing Turing computation in a foreseeable future. This problem has been tackled in several ways.

The most obvious, most widely used way is *floating point computations*. The numbers are encoded by approximations which lie in a (non-translation invariant) set of, so-called, representable numbers. When any operator is applied, be it an arithmetical operation or a function (say an analytical function) of a number, the result, which is most often not representable, is approximated, in a way which is not always well specified in norms and may be implementation-dependent. During the execution of a complex numerical method, accumulated rounding-off errors can create convergence and stability problems for the method, even if the mathematical method is proved convergent, and stable with respect to initial values. But most of all, it is generally very difficult to prove that the result is correct with such methods.

Interval Analysis (see [1] for instance) consists in performing numerical computation by applying operators to interval, rather than to (encoding of) numbers. Each operation on an interval consists of operations, or possibly bounds calculation

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which are representable and calculable, for the boundaries of the interval. In this way, the technique allows to obtain bounds for solutions of equations or numerical problems. In many cases, however, the estimations are too loose to be really useful.

Exact Real arithmetic (see an introduction in [22]) rests on the notion of a *computable number*, which consists in restricting constructions of real numbers by completion of the set $\mathbb{Q}$ of rational numbers (by Cauchy Sequences, or by decreasing sequences of intervals), narrowing the construction to calculable (through some Turing Complete language) sequences. We can define an arithmetic on the sets of numbers which, under some restrictions related to computability, implement exactly numerical computation on calculable real numbers.

Symbolic Calculus consists in formal manipulations algebraic expressions, by rewriting formally the expressions according to the rules of the algebraic structures involved. This approach to exact real computation is generally considered a natural complement to exact real arithmetic, which can fill in gaps leaved by non-computable operations. This approach, however, is quite remote from numerical calculus as we purpose to do.

In *our approach*, we would define (hopefully) any numerical problem in a discrete form and in a continuous form. The discrete form of the problems would involve only integers (with some constant denominator or scale factor). Each of the considered operators would possibly change the rational scale factor (which can be stored), but otherwise be computable using only integers. Then, a *multigrid convergence* result would allow us to show that the integer-only computable solution of the discrete problem converges to the real valued solution of the continuous (and original) version of the problem. This avoids the accumulation of rounding-off errors typical of floating point computations.

To make such an approach successful, we would need to redefine so-called digital, or discrete versions of all the objects involved in real analysis. Having made a short wish list (quadratic optimization, gradient method, *PDE* integration, and so on), we found that the notion of a derivative is at the root of all these notions. *In this paper*, we propose a general definition of a digital derivative, and prove some multigrid convergence results.

In the framework of image and signal processing, as well as shape analysis, a common problem is to estimate derivatives of functions, or tangents and curvatures of curves and surfaces, when only some (possibly noisy) sampling of the function or curve is available from acquisition. This problem has been investigated through finite difference methods, scale-space [14,16], and digital geometry [3,2,12,17,18,15,19,21,5]. Comparison of various methods is available in [13]. In our previous work [7,6,20,10,11] an approach to derivative estimation from digitized data was introduced. As in scale-space approaches, this approach is based on simple computations of *convolutions*. However, unlike scale-space methods, this approach is oriented towards integer-only models and algorithms, and is based on a discrete approach to analysis on $\mathbb{Z}$. Implementation of the convolution-based approach is straightforward. As far as the speed of convergence is concerned, the method for tangents of [20] was proved in [7] uniform worst-case $O(h^{2/3})$ for C^3 functions, where h is the size of the pixel. Moreover, our estimator allowed some (uniformly bounded or stochastic) noise. Furthermore, the method allowed to have a convergent estimation of higher order derivatives, and in particular a uniform $O(h^{4/9})$ estimation for the curvature of a generic curve.

The weakness of these results was the high complexity of computation. In this paper, we prove that the same convergence rates are allowed with a very low complexity by using sparse masks. This idea was previously used in connection with Taylor optimal masks [6]. It turns out that it is more efficient with sparse Binomial masks.

1.2. Outline

The paper is organized as follows. In Section 2, we present the notion of a digitization of a real-valued function, in which a discrete sequence of sample values approximates a real function. We also present a few noise models, including basic quantization, uniform noise or bias (depending on the scale), and stochastic noise.

In Section 3, we define digital derivatives. This notion is based on digital convolutions with masks which satisfy some identities involving sums (so-called moments). We also define some important characteristics, such as the convergence order of the mask. We also show some good properties, notably related to composition of digital derivatives by convolving the corresponding masks.

In Section 4, we begin to investigate errors for the purpose of proving multigrid convergence results. We distinguish the *sampling error*, which is due to the fact that we know only the values of the functions at a (locally) finite set of samples, from the *values error*, which is due to inaccurate input values (due to quantization, noise or bias). We also give upper bounds for both types of errors, for all the considered types of noise models.

In Section 5, we study some smoothing masks, i.e. some 0-derivative masks. These masks are important because, basically, any 1-derivative mask can be seen as a smoothing mask composed with a usual simple finite difference mask. The intuitive idea is that if we use the simple finite difference mask on noisy data, the importance of noise will be very detrimental to the results. By smoothing, which is nothing but a weighed average of values of the function, some of the noise cancels out (at least for a stochastic noise with expected value 0). We present three families of masks which we consider important, among the many masks on which we worked until now ([7,6,20], and [11] in which we began to seek low computational complexity for convolutions). These three families of masks are the Binomial, Taylor-Optimal, and B-spline masks.

In Section 6, we establish multigrid convergence results of the digital derivatives toward the corresponding continuous derivatives of the original function, providing in addition a worst case rate of convergence. That section contains the main convergence results of this paper, namely uniform convergence results in the uniform noise or bias model, and convergence results of the standard deviation of our estimation in the stochastic model. Both convergence results, which hold for