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An output-sensitive algorithm to compute the normal vector of a digital plane ☆

Jacques-Olivier Lachaud [a], Xavier Provençal [a,*], Tristan Roussillon [b]

[a] Université de Savoie, CNRS, LAMA, UMR 5127, F-73776, France
[b] Université de Lyon, CNRS, INSA-LYON, LIRIS, UMR 5205, F-69622, France

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ABSTRACT

A digital plane is the set of integer points located between the parallel planes. We solve the following problem: how to compute the exact normal vector of a digital plane given only a predicate that answers the question "is a point x in the digital plane or not". Our approach is iterative and "as local as possible". We provide a worst-case complexity bound in $\mathcal{O}(\omega \log \omega)$ calls to the predicate, where ω is equal to the arithmetic thickness parameter of the digital plane. Furthermore, our algorithm presents a much better average behavior in practice.

1. Introduction

The study of linear structures onto digital objects is of central importance in digital geometry. The first step was of course to examine 2D digital straightness (e.g. see the review of Klette and Rosenfeld [24]). These studies showed that digital straight lines have lots of properties. Their arithmetic definition as diophantine inequalities leads to optimal online recognition algorithm [11]. Their combinatorics present lots of self-symmetries [7,2] and palindromes [2,5]. Their recursive nature is directly related to the simple continued fraction approximation of the line slope [1,35]. Geometrically, digital straight lines are both digitally convex and concave [31]. The structure of their Delaunay triangulation is also related to their recursive structure [30]. These fundamental studies on digital straightness were used for analyzing 2D digital contours, by considering the finite pieces of digital straight lines included in the contours [33,18]. The ones that were not included into any one else were called *fundamental* [13] or *maximal segments* [10]. They emerge as an essential tool for analyzing digitized curves, e.g. to decompose a curve into convex and concave parts [17,13,31]. Their asymptotic properties [10] were the main ingredient for showing the multigrid convergence of several discrete estimators (tangent and length estimation [27], even curvature estimation [21,28]). They prove also to be useful to analyze the meaningful scale at which a contour should be considered and therefore to determine the importance of the noise level [22].

Hence, similar hopes are put into the study 3D linear structures, often called digital planes (e.g. see [6]), in order to tackle the problem of 3D digital shape analysis. Their arithmetic definition has been less fruitful for designing recognition

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* Corresponding author.
E-mail addresses: jacques-olivier.lachaud@univ-savoie.fr (J.-O. Lachaud), xavier.provencal@univ-savoie.fr (X. Provençal), tristan.roussillon@insa-lyon.fr (T. Roussillon).

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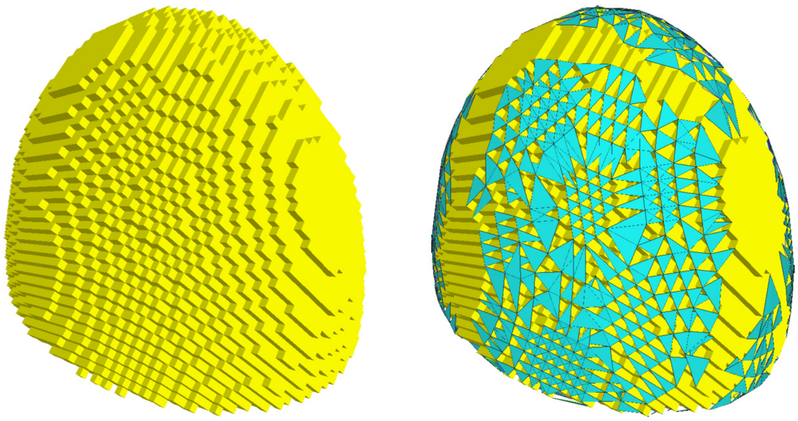

Fig. 1. Even though this paper focuses on the computation of the normal vector of an infinite digital plane, the motivations come from 3D shape analysis (See Fig. 2 and Fig. 7 for detailed examples on a digital plane.) In this image, algorithm FindNormal (Algorithm 2) is used to identify linear parts on the surface of a digitally convex object. Left: the input object is a 3D digital image described as a set of voxels. The predicate "*Is* $\mathbf{x} \in P$" is the characteristic function of the set of voxels' vertices that belong to the boundary of the digital shape. Right: algorithm FindNormal is run starting from each reentrant corner of this set. The triangles show the upper face of the output tetrahedrons.

algorithm, except for some ideas developed in [12]. Most algorithms recognizing pieces of digital planes [23,34,19] use computational geometry techniques based on convexity properties. Some of them use arithmetic but only to simplify the search of the plane parameters [8]. Their combinatorial structure has appeared to be more complex as expected [26]. This fact became apparent when studying their connectedness [20,15,4,14]. The first link between a digital plane and the two-dimensional continued fraction of its normal vector was exhibited in [16,3], and lead to a desubstitution algorithm for recognizing (specific) pieces of digital planes. It is clear that having the notion of maximal planes, natural 3D extension of maximal segments, would be extremely interesting for analyzing 3D digital surfaces. However, most works propose only a greedy segmentation into pieces of planes [25,29]. A notable exception is the empirical approach of [9] which defines maximal planes as planar extension of maximal disks.

This work proposes a new algorithm to determine the normal vector $\mathbf{N}$ of a digital plane P, given only the predicate "Is $\mathbf{x} \in P$?" where $\mathbf{x}$ is any point of $\mathbb{Z}^3$, and a starting point $\mathbf{o}$ in the plane. This algorithm is local in the sense that a few points are progressively tested around the starting point $\mathbf{o}$. This algorithm is fast since it will not check all points surrounding $\mathbf{o}$. This algorithm is simple to implement, since it requires only a few elementary tests and additions of vectors. We prove that our algorithm extracts the exact characteristics of P (i.e., integer normal vector and integer offset). Furthermore, its worst-case time complexity is essentially some $O(\|\mathbf{N}\|_1 \log \|\mathbf{N}\|_1)$. Even better, the algorithm returns also the smallest lattice basis that generates the digital plane.

This algorithm sheds new light on the relation between the geometric and combinatoric properties of digital planes and three-dimensional continued fractions. Contrary to most of multidimensional continued fraction algorithms, our algorithm does not iteratively apply the same operation, but chooses at each step, among a set of possible operations, the best one with respect to the geometry of the digital plane. It extracts progressively independent Bezout vectors for the sought normal. Moreover, the specific Delaunay structure of the digital plane is also exploited to force the algorithm to be as local as possible. This is why it can extract the smallest lattice basis of the plane. In opposition with usual plane recognition algorithms [23,34,19,8,16] whose input is a set of already identified points, this algorithm decides on the fly which next points should be tested with "Is $\mathbf{x} \in P$". Hence we believe that this algorithm induces the notion of maximal pieces of digital straight plane on a digital surface. This contribution is then the first step for analyzing 3D digital shapes with linear geometry (see Fig. 1).

The paper is organized as follows. We start in Section 2.1 by recalling basic notions about digital planes, we introduce the main definitions used in our algorithm and we state our main results. Since our algorithm has a local approach, Section 2.2 details what is the set of tested points at each step, and how they are classified into a few configurations. Then Section 2.3 and Section 2.4 describe the operations that transform progressively the initial guess. Each operation is triggered by some configuration. The whole algorithm is presented in Section 2.5. Its correctness and its worst-case complexity are respectively established in Section 3 and 4. Then Section 5 shows how the process is kept as local as possible, using Delaunay triangulation property. Last, numerical experiments indicate that the average complexity of our algorithm is low with respect to $\|\mathbf{N}\|_1$.

2. Algorithm for plane recognition

We present a new algorithm that computes the normal vector $\mathbf{N}$ of a digital plane

$$P = \{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \mathbf{x}, \mathbf{N} \rangle < \omega\},$$