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On bounded additivity in discrete tomography

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ABSTRACT

In this paper we investigate bounded additivity in Discrete Tomography. This notion has been previously introduced in [5], as a generalization of the original one in [11], which was given in terms of *ridge functions*. We exploit results from [6–8] to deal with bounded S non-additive sets of uniqueness, where $S \subset \mathbb{Z}^n$ contains d coordinate directions $\{e_1, \dots, e_d\}$, $|S| = d + 1$, and $n \geq d \geq 3$. We prove that, when the union of two special subsets of $\{e_1, \dots, e_d\}$ has cardinality $k = n$, then bounded S non-additive sets of uniqueness are confined in a grid $\mathcal{A}$ having a suitable fixed size in each coordinate direction e_i, whereas, if $k < n$, the grid $\mathcal{A}$ can be arbitrarily large in each coordinate direction e_i, where $i > k$. The subclass of pure bounded S non-additive sets plays a special role. We also compute explicitly the proportion of bounded S non-additive sets of uniqueness w.r.t. those additive, as well as w.r.t. the S-unique sets. This confirms a conjecture proposed by Fishburn et al. in [14] for the class of bounded sets.

1. Introduction

In Discrete Tomography additive sets play an important role, since the reconstruction and uniqueness problems on this class of sets can be solved in polynomial time by linear programming (whereas in general the problems are NP-hard [2]) and the additivity property can be checked efficiently [1,13]. This notion can be introduced in different ways. The original one, in [11], is given in terms of *ridge functions*, namely, a set $E \subset \mathbb{R}^n$ is said to be *additive* if, for all $i \in \{1, \dots, n\}$ there exist bounded measurable functions $f_i : \mathbb{R} \to \mathbb{R}$, such that

$$E = \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} f_i(x_i) \geq 0 \right\}.$$

Therefore, in its early beginning, additivity was an *intrinsic* property of a continuous set E, independently of any special selected set of directions. In [12] the same authors extended additivity to the discrete case, just assuming the real valued functions f_i to be defined on a discrete set. In the same paper it was also introduced the notion of *weakly bad configuration* for a set E, and proved that E is additive if and only if it has no weakly bad configurations. A weakly bad configuration for E is a pair of lattice sets (Z, W), $Z \subset E$, $W \cap E = \emptyset$, each consisting of $k \geq 2$ lattice points not necessarily distinct (counted with multiplicity), such that for each point of Z there is a point of W along all the coordinate directions. Therefore, as

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X-rays count the number of points on lines parallel to the coordinate directions, it is clear the connection and interest that additivity can assume for tomographic problems. In particular, additivity of a set E has been proved to be equivalent to the absence of weakly bad configurations for E, and implies uniqueness of reconstruction of E from the coordinate X-rays. As additivity is stronger than uniqueness, in [14] Fishburn et al. suggest that for some set of X-ray directions of cardinality larger than two, the proportion of lattice sets E of uniqueness that are not also additive approaches 1 as E gets large. They leave it as an open question in the discussion section. In the same work, the notion of additivity has been extended, and defined with respect to a *Radon base* S, consisting of linear subspaces of $\mathbb{R}^n$. This reflects in a more general definition of weakly bad configurations, where all translates of any linear subspace belonging to the Radon base S meet Z and W in the same number of points.

When a Radon base S consists only of lines, it does not necessarily contain the coordinate directions. It is often implicitly assumed that S still spans $\mathbb{Z}^n$, so that S-weakly bad configurations have full dimension, like in the original case introduced in [11]. However, S could span a proper d-dimensional subspace H of $\mathbb{Z}^n$. In this case an S-weakly bad configuration (Z, W) for a set $E \subset \mathbb{Z}^n$ assumes a more general structure. Indeed, it suffices that $Z \subset (E \cap H)$ and $W \cap E \cap H = \emptyset$.

More recently in [15], additivity has been extended to give a more general treatment of known concepts and results.

Thanks to this new approach, the authors showed that there are non-additive lattice sets in $\mathbb{Z}^3$ which are uniquely determined by their X-rays in the three standard coordinate directions by exhibiting a counter-example (see [15, Remark 2 and Figure 2]). This answers in the negative a question raised by Kuba at a conference on discrete tomography in Dagsthul (1997), that every subset E of $\mathbb{Z}^3$ might be uniquely determined by its X-rays in the three standard unit directions of $\mathbb{Z}^3$ if and only if E is additive.

1.1. Bounded additivity and new results

A further generalization of additivity is obtained by restricting to a finite lattice *tomographic grid* $\mathcal{G} \subset \mathbb{Z}^n$. This is a finite set of lattice points which are the intersection of lines parallel to the directions in S. Feasible solutions of the reconstruction problem are subsets of $\mathcal{G}$, and corresponding to nonzero X-rays in the directions in S. When S contains the coordinate directions, then the tomographic grid is an orthogonal box $\mathcal{A}$, and we simply call it *lattice grid*. In this case it becomes quite convenient to exploit the algebraic approach to DT introduced by L. Hajdu and R. Tijdeman in [17]. In [3,4] special results obtained in [16], and concerning the unique determinations of bounded sets by four X-rays, have been extended to whole families of four lattice directions. This led to the notion of *bounded S-additive* sets (and of *bounded S non-additive* sets), introduced in [5], and later extensively investigated in [6–8], with a main focus on bounded S non-additive sets of uniqueness. See also [9], where these notions provide a theoretical model for treating ghost artifacts in digital imaging from an algebraic point of view.

In the two dimensional case we proved that, when S is a minimal set of uniqueness for the lattice grid, then the proportion of bounded S non-additive sets of uniqueness w.r.t. the additive ones is constant and does not depend on the size of the grid [7].

In this paper, we completely settle the problem in $\mathbb{Z}^n$ ($n \geq 2$) by considering the coordinate directions as follows: When S spans the whole dimension n, and the union of two special subsets of S has cardinality $k = n$, then the result in [7] can be extended to any dimension (see Theorem 4). As an immediate consequence, we provide a deeper answer, in the negative, to the question raised by Kuba. Differently, for example when S spans a proper d-dimensional subspace H of $\mathbb{Z}^n$, with $d < n$, then we can distinguish "pure" bounded non-additive sets (defined as those sets E for which $E = Z$ for some S-weakly bad configuration (Z, W)) counting weakly bad configurations as in [8][1] and bounded non-additive sets. Whereas in the former case, the number of pure bounded non-additive sets is small compared to all the others, in the latter case, Theorem 8 and Theorem 12 confirm that the conjecture of Fishburn et al. holds true.

2. Preliminaries

For the sake of completeness, we wish to recall some basic definitions and results already presented in [8, Section 2].

The standard orthonormal basis for $\mathbb{Z}^n$ will be $\{e_1, \ldots, e_n\}$, and the coordinates with respect to this orthonormal basis $x_1, \ldots, x_n$. A vector $u = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, where $a_1 \geq 0$, is said to be a *lattice direction*, if $\gcd(a_1, \ldots, a_n) = 1$. We refer to a finite subset E of $\mathbb{Z}^n$ as a *lattice set*, and we denote its cardinality by $|E|$. For a finite set $S = \{u_1, u_2, \ldots, u_m\}$ of directions in $\mathbb{Z}^n$, the *dimension* of S, denoted by $\dim S$, is the dimension of the vector space generated by the vectors $u_1, u_2, \ldots, u_m$. Moreover, for each $I \subseteq S$, we denote $u(I) = \sum_{u \in I} u$, with $u(\emptyset) = 0 \in \mathbb{Z}^n$. Given a lattice direction u, the X-ray of a lattice set E in the direction u counts the number of points in E on each line parallel to u. Any two lattice sets E and F are *tomographically equivalent* if they have the same X-rays along the directions in S. Conversely, a lattice set E is said to be *S-unique* if there is no lattice set F different from but tomographically equivalent to E.

An *S-weakly bad configuration* is a pair of lattice sets (Z, W) each consisting of k lattice points not necessarily distinct (counted with multiplicity), $z_1, \ldots, z_k \in Z$ and $w_1, \ldots, w_k \in W$ such that for each direction $u \in S$, and for each $z_r \in Z$, the

[1] The estimates presented in [8] (Subsections 3.2 and 3.3) concern pure bounded S non-additive sets. Even if the computations were clearly inspired by the conjecture in [14], we wish to remark that we referred the denominator to *additive sets*, instead of *S-unique sets*, as required.