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Geometric properties of matrices induced by pattern avoidance

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ABSTRACT

The notion of submatrix avoidance in polyominoes has recently been introduced in [2] with the aim of extending most of the concepts and properties concerning pattern avoiding permutations to the setting of polyominoes. In this paper we use submatrix avoidance to describe families of polyominoes which, in the literature, are usually defined by means of the geometric constraints of *convexity*, *k-convexity*, and *directedness*. To reach this goal, we provide an extension of the notion of pattern in a polyomino, by introducing generalized polyomino patterns. In the second part of the paper, we tackle the same problem in the context of discrete sets, which can be naturally regarded as binary matrices. In this case, we consider two types of geometric constraints: *convexity* and *directedness*, and we study how these constraints can be imposed on matrices by using submatrix avoidance.

1. Introduction

Among the recently developed approaches to the study of combinatorial structures, one consists in describing them by means of the presence/absence of *patterns*. The notion of pattern has been considered in various human activities since ancient times, and research on patterns in combinatorial structures started with the study of pattern in words that dates back at least at the beginning of the 20th century with the works of Axel Thue [24,25]. Later, the research on patterns in combinatorial structures has concerned patterns in permutations [20], while in the last few years the notion of pattern has been defined and studied in other combinatorial objects such as set partitions [19,23], trees [21].

The work [2] fits into this research line with the introduction and the study of the concept of *pattern* in *polyominoes*.

Let us recall that a cell in the plane $\mathbb{Z} \times \mathbb{Z}$ is a unit square, and a *polyomino* is a finite union of cells that is connected and has no cut point (*i.e.* the set of cells has to be connected according to the edge adjacency). Polyominoes are defined up to translation (see Fig. 1). In this paper we use a quite common representation of discrete sets and polyominoes as binary matrices, where a 1 (resp. 0) entry stands for the presence (resp. absence) of a cell.

Polyominoes are popular combinatorial objects, introduced by S. Golomb [16], related to problems arising from different areas of mathematics, such as for instance: tiling problems [4,17], recreational mathematics and games [15].

The main objective of [2] is to extend most of the concepts and properties on pattern avoiding permutations to the setting of polyominoes. In particular, algebraic tools are employed in order to provide a unified framework to describe and

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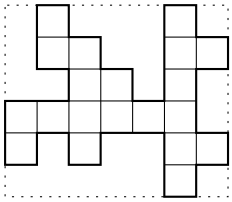

(a)

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(b)

Fig. 1. A polyomino and its representation as a binary picture (or matrix).

to handle some known families of polyominoes, by the avoidance of patterns. Therefore, in order to fruitfully present our paper in Section 2 we recall some definitions and the main results from [2].

This paper originates from [1], presented at the *18th International Conference on Discrete Geometry for Computer Imagery* (2014). The principal purpose of this paper is to extend the studies of [1] and use the notion of submatrix avoidance in order to describe families of polyominoes defined by means of geometric constraints or combinatorial properties. More precisely we investigate the problem of representing the geometric constraints of *convexity*, *k-convexity*, and *directedness* in polyominoes by combining the avoidance of five matrices, denoted by H, V, D, L_1, L_2. To reach this goal, we have run across the following related problems:

i) given a set of patterns $\mathcal{M}$, to study the class of polyominoes avoiding the patterns of $\mathcal{M}$ as submatrices, and to give a characterization of this class in terms of the geometric properties of its elements.
ii) to extend the notion of pattern in a polyomino, by introducing *generalized polyomino patterns*, in a way that is possible to describe further families of polyominoes known in the literature. Such a generalization resembles what was done for pattern avoiding permutations by the introduction of *vincular, bivincular* patterns [5].

In the second part of the paper we study the same kind of problems in the context of discrete sets, which can be naturally regarded as binary matrices. In this case, we consider two types of geometric constraints: *convexity* and *directedness*, and we study how these constraints can be imposed on matrices by using the avoidance of some of the patterns H, V, D, L_1, L_2 used for polyominoes. In the authors' opinion this research guideline gives a new insight into the study of discrete sets, in particular it lets us tackle the longstanding problem of describing sets connectedness by using this new notion of pattern avoidance.

2. Polyomino classes

Throughout all the paper we use the representation of discrete sets (resp. polyominoes) as binary matrices, understanding that the matrix has a 1 entry if there is a cell of the set (resp. polyomino) in the corresponding position, 0 otherwise. Notice that in a binary matrix representing a polyomino both the first and the last rows (resp. columns) should contain at least a 1. In this section we recall some basic definitions and results from [2], which is useful in the rest of the paper.

Definition 1. Let $\mathfrak{M}$ be the class of *matrices*. We denote by $\preccurlyeq$ the usual *submatrix* order on $\mathfrak{M}$, *i.e.* $M' \preccurlyeq M$ if M' may be obtained from M by deleting any collection of rows and/or columns.

Definition 2. Let $\preccurlyeq_P$ be the restriction of the submatrix order $\preccurlyeq$ on the set of polyominoes $\mathfrak{P}$.

This defines the poset $(\mathfrak{P}, \preccurlyeq_P)$ and the pattern order on polyominoes: a polyomino P is a *pattern* of a polyomino Q (which we denote $P \preccurlyeq_P Q$) if the binary picture representing P is a submatrix of the one representing Q. We point out that the order $\preccurlyeq_P$ has already been studied by the name of *subpicture order* in [8] in which the authors proved that $(\mathfrak{P}, \preccurlyeq_P)$ contains infinite antichains, and it is a graded poset (the rank function being the semi-perimeter of the bounding box of the polyominoes).

This allows to introduce a natural analogue of permutation classes for polyominoes:

Definition 3. A *polyomino class* is a set of polyominoes $\mathcal{C}$ that is downward closed for $\preccurlyeq_P$: for all polyominoes P and Q, if $P \in \mathcal{C}$ and $Q \preccurlyeq_P P$, then $Q \in \mathcal{C}$.

We point out that a polyomino class is just an *ideal* in the poset $(\mathfrak{P}, \preccurlyeq_P)$.

In [2] the authors established that some known families of polyominoes, including *convex*, *column-convex*, *L-convex*, *directed-convex* polyominoes, are polyomino classes. On the other side, there are several families of polyominoes, which are not polyomino classes, such as the family of polyominoes having a square shape or the family of polyominoes having exactly $k > 1$ columns.