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Note

Planar graph is on fire

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ABSTRACT

Let G be any connected graph on n vertices, $n \geq 2$. Let k be any positive integer. Suppose that a fire breaks out at some vertex of G. Then, in each turn k firefighters can protect vertices of G — each can protect one vertex not yet on fire; Next the fire spreads to all unprotected neighbours.
The *k-surviving* rate of G, denoted by $\rho_k(G)$, is the expected fraction of vertices that can be saved from the fire by k firefighters, provided that the starting vertex is chosen uniformly at random. In this paper, it is shown that for any planar graph G we have $\rho_3(G) \geq \frac{2}{21}$. Moreover, 3 firefighters are needed for the first step only; after that it is enough to have 2 firefighters per each round. This result significantly improves the known solutions to a problem by Cai and Wang (there was no positive bound known for the surviving rate of general planar graph with only 3 firefighters). The proof is done using the separator theorem for planar graphs.

1. Introduction

The following *Firefighter Problem* was introduced by Hartnell [6]. Consider any connected graph, say G, on n vertices, $n \geq 2$. Let k be any positive integer. Suppose that a fire breaks out at some vertex $v \in V(G)$. Then in each turn firefighters can protect k vertices of G, not yet on fire and the protection is permanent. Next the fire spreads to all the unprotected vertices that are adjacent to some vertices already on fire. The goal is to save as much as possible and the question is how many vertices can be saved. We would like to refer the reader to the survey by Finbow and MacGillivray [3] for more information on the background of the problem and directions of its consideration.

In this paper we focus on the following aspect of the problem. Let $sn_k(G, v)$ denote the maximum number of vertices of G that k firefighters can save when the fire breaks out at the vertex v. This parameter may depend heavily on the choice of the starting vertex v, for example when the graph G is a star. Therefore Cai and Wang [1] introduced the following graph parameter: the *k-surviving rate* $\rho_k(G)$ is the expected fraction of vertices that can be saved by k firefighters, provided that the starting vertex is chosen uniformly at random. Namely

$$\rho_k(G) = \frac{1}{|V(G)|^2} \sum_{v \in V(G)} sn_k(G, v).$$

While discussing the surviving rate, let us mention the recent results by Prałat [10,11] which have provided a threshold for the average degree of general graphs, which guarantees a positive surviving rate with a given number of firefighters. To be more precise, for $k \in \mathbb{N}^+$ let us define

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$$\tau_k = \begin{cases} \frac{30}{11} & \text{for} \quad k = 1 \\ k + 2 - \frac{1}{k+2} & \text{for} \quad k \geq 2. \end{cases}$$

Then, there exists a constant $c > 0$, such that for any $\epsilon > 0$, any $n \in \mathbb{N}^+$ and any graph G on n vertices and at most $(\tau_k - \epsilon)n/2$ edges one has $\rho_k(G) > c \cdot \epsilon > 0$. Moreover, there exists a family of graphs with the average degree tending to τ_k and the k-surviving rate tending to 0, which shows that the above result is the best possible.

The k-surviving rate is investigated for many particular families of graphs — we focus here on planar graphs. Cai and Wang [1] asked about the minimum number of firefighters k such that $\rho_k(G) > c$ for some positive constant c and any planar graph G. It is easy to see that $\rho_1(K_{2,n}) \xrightarrow[n\to\infty]{} 0$, hence at least 2 firefighters are necessary. It is shown that 2 is the upper bound for triangle-free planar graphs [2] and planar graphs without 4-cycles [8].

So far, for the general planar graphs the best known upper bound for the number of firefighters is 4: Kong, Wang and Zhu [7] have shown that $\rho_4(G) > \frac{1}{9}$ for any planar graph G. Esperet, van den Heuvel, Maffray and Sipma [2] have shown that using 4 firefighters in the first round only and just 3 in the subsequent rounds it is also possible to save a positive fraction of any planar graph G, namely $\rho_{4,3}(G) > \frac{1}{2712}$. We use the notation of $\rho_{k,l}$ and $sn_{k,l}$ to describe the model with k firefighters in the first round and l firefighters in the subsequent rounds.

In this paper, we improve the above bounds by the following theorem:

Theorem 1.1. *Let G be any planar graph. Then:*

(i) $\rho_{4,2}(G) > \frac{2}{9}$.
(ii) $\rho_3(G) \geq \rho_{3,2}(G) > \frac{2}{21}$.

In other words, we show that with 3 firefighters in the first round and just 2 in the subsequent rounds we can save at least $\frac{2}{21}$ vertices of a planar graph, while with one extra firefighter in the first round we can increase the saved fraction to $\frac{2}{9}$.

2. The proof

The proof is done using the lemma given by Lipton and Tarjan to prove the separator theorem for planar graphs [9]. The key lemma in their proof, slightly reformulated to use in the firefighter problem, is quoted below.

Lemma 2.1. *Let G be any n-vertex plane triangulation and T be any spanning tree of G. Then there exists an edge $e \in E(G) \setminus E(T)$ such that the only cycle C in $T + e$ has the property that the number of vertices inside C as well as outside C is lower than $\frac{2}{3}n$.*

A similar approach — using the above lemma to the firefighter problem on planar graphs, was first applied by Floderus, Lingas and Persson [5], with a slightly different notation of approximation algorithms. The authors of [5] have proved a theorem analogous to Lemma 2.2. For some more details see Section 3.

The proof of Theorem 1.1 is presented in two steps — first we show that $\rho_{4,2}(G) > \frac{2}{9}$ for any planar graph G, then that $\rho_{3,2}(G) > \frac{2}{21}$. At first let us note that the surviving rate is monotone (non-increasing) with respect to the operation of adding edges to the graph. Hence, it is enough to prove the bounds given by Theorem 1.1 only for plane triangulations. Moreover, in the first step, depending on the number of firefighters, we save 3 or 4 vertices respectively, which is enough to obtain the desired bounds for any planar graph on not more than 17 vertices.

Let G be any n-vertex plane triangulation, where $n \geq 18$. Suppose that the fire breaks out at a vertex r. Consider a tree T obtained by the breadth-first-search algorithm starting from the vertex r. By Lemma 2.1 there is the edge e and the cycle $C \subseteq T + e$ such that $|C \cup \mathrm{in}C| > \frac{1}{3}n$ and $|C \cup \mathrm{out}C| > \frac{1}{3}n$, where $\mathrm{in}C$ and $\mathrm{out}C$ denote the sets of vertices inside the cycle C and outside the cycle C respectively. Note that in the cycle C there are at most 2 vertices at any given distance from r. This holds because every edge of C, except one, belongs to the breadth-first-search tree. The firefighters' strategy depends on the cycle C. When the vertex r does not belong to the cycle then the firefighters protect the vertices of C in the order given by the distance from the vertex r and save all the vertices in either $C \cup \mathrm{in}C$ or $C \cup \mathrm{out}C$. When the vertex r belongs to the cycle C, the firefighters still can protect the vertices of the cycle except the vertex r, but it may be not enough, as the fire may spread through the neighbours of r inside as well as outside the cycle C. Because either $\mathrm{in}C$ or $\mathrm{out}C$ contains not more than $\left\lfloor \frac{\deg r - 2}{2} \right\rfloor$ neighbours of r we get immediately:

Lemma 2.2. *Let G be any n-vertex plane triangulation, where $n \geq 18$. Suppose that the fire breaks out at some vertex r. Then using $2 + \left\lfloor \frac{\deg r - 2}{2} \right\rfloor$ firefighters at the first step and 2 at the subsequent steps one can save more than $n/3 - 1$ vertices.*

To calculate the surviving rate $\rho_{4,2}(G)$ let us now partition the vertex set of the graph G into 3 sets: $X = \{v \in V(G) \colon \deg(v) \in \{3, 4\}\}$, $Y = \{v \in V(G) \colon \deg(v) \in \{5, 6, 7\}\}$ and $Z = \{v \in V(G) \colon \deg(v) \geq 8\}$. Obviously we have $|Z| = n - |X| - |Y|$. Since for the plane triangulation we have