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Probabilistic connectivity threshold for directional antenna widths

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A B S T R A C T

Consider the task of maintaining connectivity in a wireless network where the network nodes are equipped with directional antennas. Nodes correspond to points on the unit disk and each uses a directional antenna covering a sector of a given angle α.
The width required for a connectivity problem is to find out the necessary and sufficient conditions of α that guarantee connectivity when an antenna's location is uniformly distributed and the orientation of the antenna's sector is either random or fixed.
We show that when the number of network nodes is big enough, the required $\breve{\alpha}$ approaches zero. Specifically, on the unit disk, assuming uniform orientation, it holds with high probability that the threshold for connectivity is $\breve{\alpha} = \Theta(\sqrt[4]{\frac{\log n}{n}})$. This is shown by the use of Poisson approximation and geometrical considerations. Moreover, when the model is relaxed, assuming that the antenna's orientation is directed towards the center of the disk, we demonstrate that $\breve{\alpha} = \Theta(\frac{\log n}{n})$ is a necessary and sufficient condition.

1. Introduction

Communication among wireless devices is of great interest in the current wireless technology, where devices are part of sensor networks, mobile ad-hoc networks and RFID devices that take part in the emerging ubiquitous computing, and even satellite networks. These communication networks are usually extremely dynamic, where devices frequently join and leave (or crash) and, therefore, require probabilistic techniques and analysis. Imagine, for example, sensor networks that use directional antennas (saving energy and increasing communication capacity) among the sensors that should be connected even though they are deployed by an airplane that drops them from the air (just as in a smart dust scenario). What is the density of those sensors needed to ensure their connectivity? Is there a way to renew connectivity after some portion of the sensors stop functioning—maybe by deploying only an additional fraction, uniformly distributed in the area with random orientation of the antennas? In this work, we try for the first time to suggest and analyze ways to ensure connectivity in such probabilistic scenarios. Namely, we have studied the problem of arranging randomly scattered wireless sensor antennas in a way that guarantees the connectivity of the induced communication graph. The main challenge here is to minimize energy consumption while preserving node connectivity.

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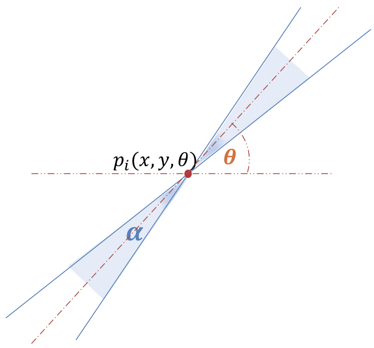

Fig. 1. Fixing at each point p_i two opposite wedges of angle α with direction θ.

In order to save power, increase transmission capacity and reduce interference [9], the antennas do not communicate information in all directions but rather inside a wedge-shaped area. Namely, *the coverage area* of a directional antenna located at point p of angle α is two opposite sectors of angle α of the unit disk centered at p (see Fig. 1). Throughout the paper we will call α *the communication angle*.

The smaller the angle is, the better it is in terms of energy saving. When knowing nothing about the future positioning of the antennas, each antenna may be directed in a random direction that may stay fixed forever. Therefore, we wish to find the minimum $\alpha > 0$ so that no matter what finite set of locations the antennas are given, with high probability they can communicate with each other. Our goal is to specify necessary and sufficient conditions for the width of wireless antennas that enable one to build a connected communication network when antenna's locations and directions are randomly and uniformly chosen.

Throughout this paper, we refer to an undirected graph where the nodes are the antennas, and two nodes are connected by an edge if and only if their corresponding antennas are located in each other's transmission area. However, our calculations hold for the directed case as well. Specifically, Theorems 2 and 3 hold for both cases, and the result proven by Theorems 4 and 5 also implies a connectivity threshold for the directed graph case.

Previous results that handle wireless directional networks [4,1] assume coordinated locations and orientations for the antennas. They show that a connected network can be built with antennas of width $\alpha = \pi/3$. The same model's assumptions were used by [2] to study graph connectivity in the presence of interference and in [8] to optimize the transmission range as well as the hop-stretch factor of the communication network. A different model of a *directed* graph of directional antennas of bounded transmission range was studied in [3,5].

In contrast to the above worst case approaches, to the best of our knowledge, we consider for the first time the connectivity problem from a probabilistic perspective. Namely, we are interested in the minimal communication angle that implies *high probability* for the graph to be connected as a function of the number of nodes. This approach significantly reduces the required communication angle and is more general in the sense that we don't have to make directing procedures as the procedures employed in [4,1,2] algorithms.

The probabilistic setting of the problem is related to other research in the field of continuum percolation theory [10]. The model for the points here is a Poisson point process, and the focus is on the existence of a connected component under different models of connections. For example, [13] studied the number of neighbors that implies connectivity. Papers [12,6] are focused on the minimum number r such that two points are connected if and only if their metric distance is $\leq r$. In [7] the authors generalized the results in [12,6] and proved that for a fractal in $\mathbb{R}^d$, it holds with high probability that $r \approx (\frac{\log n}{n})^{1/d}$, where $\approx$ means that the ratio of the two quantities is between two absolute constants.

Our main results (summarized in Theorems 2, 3, 4 and 5) discuss two different models. The first is related to the case where all the antennas are directed to one reference point (specifically, we used the center of a disk). The second model generalizes the results by dealing with randomly chosen locations and directions. Assuming that the number of nodes is big enough, we show in both cases that with high probability, the threshold $\check{\alpha}$ approaches zero. We believe that these results are important for both their combinatorial and geometric perspectives and in their implications on the design of wireless networks.

2. Preliminaries

Let $P = \{p_1(x, y, \theta), \ldots, p_n(x, y, \theta)\}$ be a set of n *points* (or *nodes*). The point location (x, y) is chosen independently from the uniform distribution over the unit disk $\mathbb{D}$ (or over the unit disk boundary $\partial\mathbb{D}$) in the plane. The antenna direction θ of each point might be fixed or to be chosen independently and uniformly at random from $[0, \pi]$. Each point represents a communication station by a pair of opposite wedges of angle α with direction θ at each node (see Fig. 1).

Given nodes u and v with communication angle α, we say that u *sees* v if u lies in the coverage area (on the intercepted arc) of v.