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A characterization of eventual periodicity

Teturo Kamae [a], Dong Han Kim [b,*]

[a] Advanced Mathematics Institute, Osaka City University, Osaka, 558-8585, Japan
[b] Department of Mathematics Education, Dongguk University – Seoul, Seoul 100-715, Republic of Korea

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ABSTRACT

In this article, we show that the Kamae–Xue complexity function for an infinite sequence classifies eventual periodicity completely. We prove that an infinite binary word $x_1x_2\cdots$ is eventually periodic if and only if $\Sigma(x_1x_2\cdots x_n)/n^3$ has a positive limit, where $\Sigma(x_1x_2\cdots x_n)$ is the sum of the squares of all the numbers of occurrences of finite words in $x_1x_2\cdots x_n$, which was introduced by Kamae–Xue as a criterion of randomness in the sense that $x_1x_2\cdots x_n$ is more random if $\Sigma(x_1x_2\cdots x_n)$ is smaller. In fact, it is known that the lower limit of $\Sigma(x_1x_2\cdots x_n)/n^2$ is at least $3/2$ for any sequence $x_1x_2\cdots$, while the limit exists as $3/2$ almost surely for the $(1/2, 1/2)$ product measure. For the other extreme, the upper limit of $\Sigma(x_1x_2\cdots x_n)/n^3$ is bounded by $1/3$. There are sequences which are not eventually periodic but the lower limit of $\Sigma(x_1x_2\cdots x_n)/n^3$ is positive, while the limit does not exist.

1. Introduction

In [1], a criterion of randomness for binary words is introduced. As stated in Definitions 1 and 3, let

$$\Sigma(x_1x_2\cdots x_n) = \sum_{\xi\in\cup_{k=1}^{\infty}\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2,$$

where

$$|x_1x_2\cdots x_n|_\xi := \#\{i : 1 \le i \le n-k+1,\ x_ix_{i+1}\cdots x_{i+k-1} = \xi\}$$

is the number of occurrences of a finite word ξ in $x_1x_2\cdots x_n$. Since the function $f(x) = x^2$ is convex, the value $\sum_{\xi\in\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2$ for any $k = 1, 2, \cdots$ is smaller if the values $|x_1x_2\cdots x_n|_\xi$ for $\xi \in \{0, 1\}^k$ deviate less as a whole from the mean value $(n-k+1)/2^k$, that is, the sequence $x_1x_2\cdots x_n$ is more random. In fact, it is proved in [1] that

$$\liminf_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^2} \ge \frac{3}{2}$$

holds for any $x_1x_2\cdots \in \{0, 1\}^\infty$, while

* Corresponding author.
E-mail addresses: kamae@apost.plala.or.jp (T. Kamae), kim2010@dongguk.edu (D.H. Kim).
URL: http://www14.plala.or.jp/kamae (T. Kamae).

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$$\lim_{n\to\infty} \frac{\Sigma(X_1X_2\cdots X_n)}{n^2} = \frac{3}{2}$$

holds with probability 1 if $X_1X_2\cdots X_n$ is the i.i.d. process with $P(X_i=0)=P(X_i=1)=1/2$.

In this article, we study the opposite case that $\Sigma(x_1x_2\cdots x_n)$ increases in the order of n^3 and prove that $x_1x_2\cdots \in \{0,1\}^\infty$ is eventually periodic if and only if

$$\lim_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} \text{ exists and } > 0.$$

It is easy to see that if $x = x_1x_2\cdots \in \{0,1\}^\infty$ contains few 1s, or precisely speaking, if $x = 0^{k_1}10^{k_2}1\cdots$ with $\liminf_{n\to\infty} k_{n+1}/k_n > 1$, then we have

$$\liminf_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} > 0.$$

Since this $x_1x_2\cdots$ is not eventually periodic, it follows from our result that $\lim_{n\to\infty}\Sigma(x_1x_2\cdots x_n)/n^3$ does not exist.

There are many characterizations of eventual periodicity. The most famous one might be the result due to Morse and Hedlund concerning the complexity. That is, $x_1x_2\cdots$ is eventually periodic if and only if for some $k\ge 1$ the number of words of size k appearing in $x_1x_2\cdots$ is smaller than $k+1$ [3]. Another characterization concerning the return time is obtained in [2]. Here, we add one more characterization which concerns both the complexity and the return time.

2. Definitions and lemmas

Definition 1. For $x_1x_2\cdots x_n \in \{0,1\}^n$, $\xi \in \{0,1\}^k$ with $1\le k\le n$ and $i=0,1,\cdots,n-k$, we denote

$$\xi \prec_i x_1x_2\cdots x_n \text{ if } \xi = x_{i+1}x_{i+2}\cdots x_{i+k}$$

and

$$\xi \prec x_1x_2\cdots x_n \text{ if } \xi \prec_i x_1x_2\cdots x_n \text{ for some } i=0,1,\cdots,n-k.$$

We call ξ a *factor* or *suffix* of $x_1x_2\cdots x_n$, respectively, if $\xi \prec x_1x_2\cdots x_n$ or $\xi \prec_{n-k} x_1x_2\cdots x_n$. We also denote

$$|x_1x_2\cdots x_n|_\xi = \#\{i : 0\le i\le n-k,\ \xi \prec_i x_1x_2\cdots x_n\}$$

and $|x_1x_2\cdots x_n| = n$.

Definition 2. For $\eta = a_1\cdots a_k \in \{0,1\}^k$ and $\ell = 1,2,\cdots$, we denote

$$\eta^\ell = \underbrace{a_1\cdots a_k}_{1}\underbrace{a_1\cdots a_k}_{2}\cdots\underbrace{a_1\cdots a_k}_{\ell}.$$

In the same way, we define $\eta^\infty \in \{0,1\}^\infty$. We call η *primitive* if there is no ξ such that $\eta = \xi^\ell$ for some $\ell\ge 2$.

Definition 3. (See [1].) Define $\Sigma^n : \{0,1\}^n \to \mathbb{R}$ by

$$\Sigma^n(x_1x_2\cdots x_n) = \sum_{\xi\in\{0,1\}^+} |x_1x_2\cdots x_n|_\xi^2,$$

where $\{0,1\}^+ = \bigcup_{k=1}^\infty \{0,1\}^k$. We write $\Sigma^n = \Sigma$ as a function from $\{0,1\}^+$ to $\mathbb{R}$.

Definition 4. For $x_1x_2\cdots x_n \in \{0,1\}^n$, define

$$\Lambda(x_1x_2\cdots x_n) = \max\{|\eta|^2(\ell+1)^3 : \eta^\ell \prec x_1x_2\cdots x_n\}$$

Lemma 1. *For any $x_1x_2\cdots x_n \in \{0,1\}^n$, it holds that*

$$\Sigma(x_1x_2\cdots x_n) \ge \frac{\Lambda(x_1x_2\cdots x_n)}{48}.$$

Proof. Let $M = \Lambda(x_1x_2\cdots x_n)$. Then, there exist positive integers k,ℓ and $\eta\in\{0,1\}^k$ with $\eta^\ell \prec x_1x_2\cdots x_n$ such that $k^2(\ell+1)^3 = M$. Then, we have

$$\sum_{\xi;\ \xi\prec\eta} |\eta^\ell|_\xi^2 \ge \ell \sum_{\xi;\ \xi\prec\eta} |\eta^\ell|_\xi \ge \frac{k^2\ell^2}{2},$$