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On the construction of all shortest vertex-disjoint paths in Cayley graphs of abelian groups

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ABSTRACT

Cayley graphs provide a group-theoretic model for designing and analyzing symmetric interconnection networks. In this paper, we give a sufficient condition for the existence of m vertex-disjoint shortest paths from one source vertex to other m (not necessarily distinct) destination vertices in a Cayley graph of an abelian group, where $m \leq n$ and n is the cardinality of a (group) generator of the abelian group. In addition, when the condition holds, the m vertex-disjoint shortest paths can be constructed in $O(mn)$ time, which is optimal in the worst case when $O(n) \leq$ the order of diameter. By applying our results, we can easily obtain the necessary and sufficient conditions, which can be verified in nearly optimal time, for the existence of all shortest vertex-disjoint paths in generalized hypercubes and odd tori. In the situation that all of the source vertex and destination vertices are mutually distinct, brute-force computations show that the probability of the existence of the m vertex-disjoint shortest paths in an r-dimensional generalized hypercube with r coordinates each of order k is greater than 94%, 70%, 96%, 99%, and 99% for $(k, r, m) = (2, 7, 7)$, $(3, 4, 8)$, $(4, 3, 6)$, $(5, 3, 6)$, and $(6, 3, 6)$, respectively.

1. Introduction

Due to advanced hardware technology, it is now feasible to build a large-scale multiprocessor system consisting of hundreds or even thousands of processors. One crucial step in designing a multiprocessor system is to determine the topology of its interconnection network (network, for short) in which nodes and links correspond to processors and communication channels, respectively. Since the network topology plays a significant role in system performance, many possible options have been proposed in the literature. Some popular network topologies include hypercube networks (hypercube, for short) [16,17,20,21,25,27], torus networks (torus, for short) [3,4,9,30], generalized hypercube networks (generalized hypercube, for short) [5,12], and star networks [1,8,23].

Routing is a process of transmitting messages among nodes/processors. Its efficiency and reliability, which are crucial to the system performance, can be enhanced by employing internally node-disjoint paths (disjoint paths, for short), because they can be used to avoid congestion, accelerate transmission rate, and provide alternative transmission routes. Two paths are *internally node-disjoint* if they do not share any common node except their end nodes. In order to reduce the transmission latency and cost, routing with disjoint paths [8,9,12,17,22,25] expects their maximal length and total length to be minimized, respectively, where the *length* of a path is the number of links in it. The benefits of disjoint paths make them play an

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Table 1
Disjoint paths of various networks.

Network	One-to-one	One-to-many	Many-to-many
Hypercube	Optimal	Optimal	Available
k-ary n-cube	Optimal	Optimal	Unavailable
Folded hypercube	Optimal	Optimal	Unavailable
Generalized hypercube	Optimal	Optimal	Unavailable
Hierarchical hypercube	Available	Available	Available
Star	Optimal	Nearly optimal	Available
(n,k)-star	Nearly optimal	Available	Unavailable
Butterfly	Optimal	Optimal	Unavailable

important role in the study of routing, reliability, and fault tolerance in parallel and distributed systems [8–13,16,17,20–23,25,26,28,29].

There are three categories of disjoint paths, i.e., one-to-one, one-to-many, and many-to-many [10]. Suppose that W is a network with connectivity n, where the *connectivity* of a network is the minimum number of nodes whose removal can make the network disconnected or trivial [7]. According to Menger's theorem [7], there exist n disjoint paths between every two distinct nodes of W. They belong to the one-to-one category. Many one-to-one disjoint paths constructed for a variety of networks can be found in the literature [9,12,13,19–21,23,28]. On the other hand, according to Theorem 2.6 in [6], there exist n disjoint paths from one node to other n distinct nodes in W. They belong to the one-to-many category. One-to-many disjoint paths were first studied in [26] where the Information Dispersal Algorithm (IDA, for short) was proposed on the hypercube. By taking advantages of disjoint paths, the IDA has numerous potential applications to secure and fault-tolerant storage and transmission of information. Some examples of one-to-many disjoint paths can be found in [8,10,11,13,17,19–22,25,26,29]. Many-to-many disjoint paths (or named set-to-set disjoint paths), which connect two sets of nodes in W, can be found in [16,17]. As shown in Table 1, optimal disjoint paths of various networks have been constructed so that their maximal length was minimized in the worst case.

In order to exploit the topological properties of a network, it was often modeled as an undirected graph in which the vertices and edges correspond to the nodes and links of the network, respectively. In [1], a group-theoretic model, named Cayley graph model, was proposed for designing and analyzing node-symmetric networks which have the property that the network viewed from any node of the network looks the same. Given a (group) generator of a finite group such that it is closed under inverse, a Cayley graph can be drawn so that the vertices correspond to the elements of the group, and there is an edge from one element to the other element if the latter element can be produced by applying group operation on the former element and an element in the generator. A *generator* of a group is a set of elements (excluding the identity) of the group so that every element of the group can be produced by applying group operation on a (finite) sequence of elements all taken from the generator. Every Cayley graph has been shown vertex transitive (i.e. node symmetric) in [1]. In addition, every node-symmetric network can be represented by this model or a simple extension of it. By the aid of this model, many properties can be proved for a class of networks, instead of each network individually [2,13–15,18,24,31].

Routing functions have been shown effective in deriving disjoint paths in the hypercube and its variations [20–22]. By the aid of routing functions, it was shown in [21] that m disjoint shortest paths from one source node to other m (not necessarily distinct) destination nodes can be constructed in an n-dimensional hypercube (n-cube, for short), provided the existence of such disjoint shortest paths which can be verified in $O(mn^{1.5})$ time (also see [25]), where $m \le n$. The time and space complexities of the construction procedure are both optimal $O(mn)$. In this paper, we study the problem of constructing m disjoint shortest paths from one source vertex to other m (not necessarily distinct) destination vertices in a Cayley graph of an abelian group, where $m \le n$ and n is the cardinality of a generator of the abelian group. Based on our idea, this problem was first transformed into a corresponding problem of constructing disjoint shortest paths with special properties in an n-cube, and then its solutions are applied to construct the required paths. By the aid of the construction procedure of [21], it will be shown that m disjoint shortest paths from one source vertex to other m (not necessarily distinct) destination vertices can be constructed in the Cayley graph if a condition for the existence of such disjoint shortest paths holds. In addition, the time and space complexities of our construction procedure are both $O(mn)$, which is optimal in the worst case when $O(n) \le$ the order of diameter of the Cayley graph.

Since both the generalized hypercube and odd torus belong to the Cayley graphs of abelian groups, the necessary and sufficient conditions, which can be verified in nearly optimal time, for the existence of all shortest disjoint paths in them can be obtained easily by applying our results. In order to figure out the probability that there exist disjoint shortest paths in a generalized hypercube, brute-force computations were carried out for verifying all the combinations of the source vertex and destination vertices by running computer programs. In the situation that all of the source vertex and destination vertices are mutually distinct, computational results show that the probability of the existence of the m disjoint shortest paths in an r-dimensional generalized hypercube with r coordinates each of order k is greater than 94%, 70%, 96%, 99%, and 99% for $(k, r, m) = (2, 7, 7)$, $(3, 4, 8)$, $(4, 3, 6)$, $(5, 3, 6)$, and $(6, 3, 6)$, respectively. It indicates that the construction of disjoint shortest paths in an r-dimensional generalized hypercube is not only theoretically interesting but also practical in real applications.