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Upward planar graphs and their duals ☆,☆☆

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ABSTRACT

We consider upward planar drawings of directed graphs in the plane (**UP**), and on standing (**SUP**) and rolling cylinders (**RUP**). In the plane and on the standing cylinder the edge curves are monotonically increasing in y-direction. On the rolling cylinder they wind unidirectionally around the cylinder. There is a strict hierarchy of classes of upward planar graphs: **UP** ⊂ **SUP** ⊂ **RUP**.

In this paper, we show that rolling and standing cylinders switch roles when considering an upward planar graph and its dual. In particular, we prove that a strongly connected graph is **RUP** if and only if its dual is a **SUP** dipole. A dipole is an acyclic graph with a single source and a single sink. All **RUP** graphs are characterized in terms of their duals using generalized dipoles. Moreover, we obtain a characterization of the primals and duals of **wSUP** graphs which are upward planar graphs on the standing cylinder and allow for horizontal edge curves.

1. Introduction

Directed graphs are used as a model for structural relations where the vertices represent entities and the edges express dependencies. Such graphs are often acyclic and are drawn as hierarchies using the framework introduced by Sugiyama et al. [30]. This drawing style transforms the edge direction into a geometric direction: all edges point upward. If only planar drawings are allowed, we obtain *upward planar graphs*, **UP** for short. These graphs can be drawn (straight-line) in the plane such that the edge curves are monotonically increasing in y-direction and do not cross. Such drawings respect a unidirectional flow of information and planarity.

Independently, Platt [29], Kelly [25], and Di Battista and Tamassia [12] characterized the upward planar graphs as the subgraphs of planar st-graphs. An *st-graph* is a directed acyclic graph with a single source s and a single sink t and the edge (s,t). The recognition problem for **UP** is $\mathcal{NP}$-hard in general [19], and it is solvable in linear time if the graphs have a single source [24] or in polynomial time if they are 3-connected [9], outer planar [11,18,28], and series-parallel [11,13].

Upward planarity on surfaces other than the plane generally deals with drawings of graphs on a fixed surface in $\mathbb{R}^3$ such that the curves of the edges are monotonically increasing in y-direction. Examples are the standing [10,20,26,27,32] and rolling cylinders [10], the sphere and the truncated sphere [15,17,21,23], and the lying and standing tori [14,16]. In full

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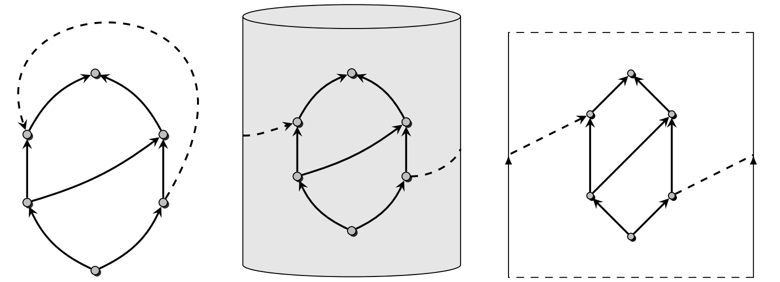
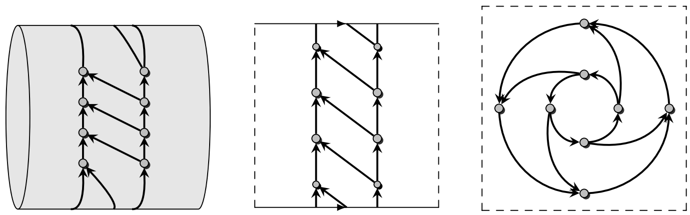


(a) An **UP** graph (without the dashed edge). (b) Its **SUP** drawing. (c) Fundamental polygon representation.

Fig. 1. An example of a **SUP** graph that is not **UP**.



(a) Rolling cylinder. (b) Fundamental polygon. (c) Cyclic representation.

Fig. 2. Different representations of a **RUP** graph.

generality upward planarity is defined on arbitrary two-dimensional manifolds endowed with a vector field prescribing the direction of the edges [2,22]. Then the standing cylinder corresponds to the plane with a radial field where the direction is away from the center, and the rolling cylinder corresponds to the plane with a concentric field where the direction is circular. Alternatively, the plane and the standing and rolling cylinders can be represented by the fundamental polygon, where the plane is identified with $I \times I$, and I is the open interval from -1 to $+1$. We obtain $I_\circ$ by identifying the boundaries of I. The *standing* (*rolling*) *cylinder* is then defined by $I_\circ \times I$ ($I \times I_\circ$), i.e., by identifying the left and right (upper and lower) boundaries of the fundamental polygon. It is well known that every undirected planar graph has a planar drawing on any surface of genus 0, such as the plane, the sphere, and the cylinder. This does no longer hold for upward planar drawings of directed graphs.

The extension of upward planarity to the sphere was addressed by Rival and his co-authors for straight-line drawings of partial orders [17,22,23]. Thomassen [32] considered such drawings on the standing cylinder. We call a graph **SUP** graph if it has a planar upward drawing on the standing cylinder. This is equivalent to upward planarity on the sphere where all edge curves are increasing from the south pole s to the north pole t. The equivalence was formally established in [2]. **SUP** graphs closely resemble **UP** graphs and were characterized as spanning subgraphs of planar dipoles [20,21,26,32]. A *dipole* is a directed acyclic graph with a single source s and a single sink t which in contrast to an st-graph does not have to contain the edge (s, t).

If there is an st-edge, then edges cannot wind around the cylinder and the sphere, which enforces that a **SUP** graph with an st-edge is **UP**. Clearly, **UP** $\subset$ **SUP**, where the graph from Fig. 1 shows the strictness of the inclusion [12,25]. The recognition of **SUP** graphs is $\mathcal{NP}$-hard [23] and is solvable in linear time for 3-connected graphs with a single source [15]. Graphs in **SUP** − **UP** need edges that wind around the cylinder, but no single edge has to make a complete winding [10]. **SUP** graphs play an important role in radial drawings [5].

Upward planar drawings on the rolling cylinder substantially differ from those in the plane and on the standing cylinder. In particular, they allow for cycles winding around the cylinder. In the fundamental polygon upward means in y-direction. Such drawings arise from recurrent hierarchies [6,30] by the restriction to planarity. An example is shown in Figs. 2(a) and 2(b). Let **RUP** denote the set of graphs with an upward plane drawing on the rolling cylinder. Visibility representations and **RUP** drawings of planar graphs were studied by Tamassia and Tollis [31]. However, there is no characterization of **RUP** graphs alike that of **UP** and **SUP** graphs. Again, the recognition problem for **RUP** graphs is $\mathcal{NP}$-hard [10], and there is a linear-time algorithm to test whether a directed graph without sources and sinks is **RUP** [1,4].

The relaxation of monotone to non-decreasing edge curves has no effect for **UP** and **RUP** graphs, since horizontal edge segments can be avoided by a lifting [10]. However, non-decreasing curves allow for horizontal cycles around the standing cylinder or the sphere. Such drawings are called *weakly standing upward planar* and the respective class of graphs is **wSUP** [2,10].