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Power and exponential sums for generalized coding systems by a measure theoretic approach

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ABSTRACT

A measure theoretic approach to study the power and the exponential sums for the usual coding system has been developed since the 1990s. In this paper, we first introduce a new coding system, and then give explicit formulas for the power and the exponential sums for the coding system by the measure theoretic approach. An expression for the power sum using the generalized Takagi function will also be given.

1. Introduction

Let $q \geq 2$ be an integer. In this paper, the notation $\mathbf{N}$ means the set of positive integers. We denote the q-adic expansion of $n \in \mathbf{N} \cup \{0\}$ by $n = \sum_{i \geq 0} n_i q^i$, where $n_i \in \{0, 1, \ldots, q-1\}$, and define the sum of digits function S_q by

$$S_q(n) = \sum_{i \geq 0} n_i.$$

There are many early works for the sums

$$\sum_{n=0}^{N-1} S_q^k(n) \quad \text{(power sum)},$$

$$\sum_{n=0}^{N-1} e^{\xi S_q(n)} \quad \text{(exponential sum)},$$

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where $N, k \in \mathbf{N}$ and $\xi \in \mathbf{R}$. If N is a power of q, we immediately have explicit formulas for these sums. However, it is not so easy to obtain such formulas for arbitrary N. For the power sum, in early stages of study, asymptotic behaviors and evaluations of error terms were performed by Bush [2] and Mirsky [23]. In the case of $k = 1$, Trollope [32] first obtained a precise formula for $q = 2$, and Delange [7] generalized it for arbitrary q with an elegant proof. In the case of $k \geq 2$, several explicit formulas were obtained by Coquet [5], Osbaldestin [28], and Grabner, Kirschenhofer, Prodinger, and Tichy [14]. For the exponential sum, in the case of $q = 2$, asymptotic behaviors and evaluations of error terms were performed by Stolarsky [31], Harborth [16], Coquet [4], and Stein [30]. Certain kinds of explicit formulas were also obtained by [30] and [31].

In the 1990s, Okada, Sekiguchi, and Shiota [25,26] studied the power and the exponential sums from a viewpoint of the measure theory and found the close relation between $S_q(n)$ and the multinomial measure on the unit interval. They introduced a new method of investigation: (i) firstly, derive an explicit formula for the exponential sum by use of the distribution function of the multinomial measure, (ii) secondly, differentiate both sides of the formula with respect to ξ, (iii) and finally, take $\xi = 0$ to get an explicit formula for the power sum. We remark that, in this method, a series of results on the Takagi function (Hata and Yamaguti [17], Sekiguchi and Shiota [29], Okada, Sekiguchi, and Shiota [27]) plays an essential role.

The power and the exponential sums for another coding systems have been also considered by several authors. The typical coding system is the reflected binary code (RBC), which is also called the Gray code (Gray [15]). The power and the exponential sums for RBC was studied by Flajolet and Ramshaw [11], Flajolet, Grabner, Kirschenhofer, Prodinger, and Tichy [12], Grabner and Tichy [13], and Kobayashi [22]. Dumont and Thomas [8–10] introduced a new coding system associated with a fixed point of a substitution and studied the power sum for it. Chen, Hwang and Zacharovas [3] reviewed probabilistic properties for the sum of digits function of random integers and gave new results on related asymptotic behaviors.

Recently, Kamiya and Murata [18] investigated the power and the exponential sums for RBC and found an interesting relation between RBC and the regular paperfolding sequence. In Kamiya and Murata [19], noticing this relation, they introduced a new coding system related to the generalized paperfolding sequence. Furthermore, for this coding system, they obtained an explicit formula for the power sum in the case of $k = 1$.

In this paper, inspired by the coding system studied in [19], we first introduce a new coding system, which includes many known coding systems as a particular case, and then, by the measure theoretic approach developed in [25,26], give explicit formulas for the power and the exponential sums for this coding system.

In Section 2, we review the generalized paperfolding sequence, which is a starting point of our research. We introduce a new coding system in Section 3 and the probability measure corresponding to it in Section 4. We next state our results in Section 5: (i) a simple explicit formula for the exponential sum (Theorem 1), (ii) an expression for the power sum using higher order derivatives of the distribution function of this measure (Theorems 2 and 3), (iii) and expressions for these derivatives using the generalized Takagi functions (Theorem 4 and Corollary 1). In Section 6, we give some examples. Sections 7 and 8 are devoted to the proofs of Theorems 1 and 2, respectively. Since the proof of Theorem 4 is very long and technical, we only sketch out the proof in Section 9 and refer to [20] for the complete one.

2. Generalized paperfolding sequences

Let us fold a strip of paper in half. As for the direction of folding, we put the right half part over the left half part. Next, let us fold the folded strip of paper in half keeping the direction of folding. Repeating this folding many times, and then unfolding, we get a sequence of 2 types of shape "$\vee$" and "$\wedge$". We assign 1 to $\vee$ and -1 to $\wedge$. Note that this assignment is different from that of [19]. Here we follow the assignment of Dekking [6]. Then we obtain the finite sequence

$$1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, \ldots.$$

This sequence is naturally extended to the infinite sequence, and it is called the regular paperfolding sequence.

Here we focus on the direction of folding. We assign 1 to "putting the right half part over the left half part", -1 to "putting the right half part under the left half part", and call this 1 or -1 the folding instruction. The regular paperfolding sequence is associated with the sequence of folding instructions $\{1, 1, 1, \ldots\}$.

Let $\boldsymbol{b} = \{b_k\}_{k=0}^{\infty}$ be a sequence of folding instructions. Then the infinite sequence associated with $\boldsymbol{b}$ can be defined (see, e.g., Allouche and Shallit [1, Section 6.5]), which is called the generalized paperfolding sequence and denoted by $\{P_{\boldsymbol{b}}(n)\}_{n=1}^{\infty}$.

In [19], it is shown that $\{P_{\boldsymbol{b}}(n)\}_{n=1}^{\infty}$ is related to a certain code in the case of periodic $\boldsymbol{b}$ with $b_0 = 1$. Let K be the period of $\boldsymbol{b}$, σ_0 the permutation

$$\sigma_0 = \begin{pmatrix} 0 & 1 & \cdots & 2^K - 2 & 2^K - 1 \\ 2^K - 1 & 2^K - 2 & \cdots & 1 & 0 \end{pmatrix},$$

and η the finite code defined by