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Coloring clique-hypergraphs of graphs with no subdivision of K_5

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ABSTRACT

A clique-coloring of a graph G is a coloring of the vertices of G so that no maximal clique of size at least two is monochromatic. The clique-hypergraph, $\mathcal{H}(G)$, of a graph G has $V(G)$ as its set of vertices and the maximal cliques of G as its hyperedges. A (vertex) coloring of $\mathcal{H}(G)$ with no monochromatic hyperedge is a clique-coloring of G. The clique-chromatic number of G is the least number of colors for which G admits a clique-coloring. Every planar graph has been proved to be 3-clique-colorable and every claw-free planar graph, different from an odd cycle, has been proved to be 2-clique-colorable. In this paper we first generalize the result of planar graphs to K_5-minor-free graphs. Furthermore, we generalize the result of claw-free planar graphs to K_5-subdivision-free graphs and give a polynomial-time algorithm to find a 2-clique-coloring of K_5-subdivision-free graphs.

1. Introduction

A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathcal{E})$ where V is a finite set of vertices and $\mathcal{E}$ is a family of non-empty subsets of V called *hyperedges*. A *k-coloring* of $\mathcal{H}$ is a function $\phi : V \to \{1, 2, \dots, k\}$ such that for each $S \in \mathcal{E}$, with $|S| \geq 2$, there exist $u, v \in S$ with $\phi(u) \neq \phi(v)$, that is, there is no monochromatic hyperedge of size at least two. If such a function exists we say that $\mathcal{H}$ is *k-colorable*. The *chromatic number* $\chi(\mathcal{H})$ of $\mathcal{H}$ is the smallest k for which $\mathcal{H}$ admits a k-coloring. In other words, a k-coloring of $\mathcal{H}$ is a partition $\mathcal{P}$ of V into at most k parts such that no hyperedge of cardinality at least 2 is contained in some $P \in \mathcal{P}$.

Here we consider hypergraphs arising from graphs: for an undirected simple graph G, we call *clique-hypergraph* of G (or *hypergraph of maximal cliques* of G) the hypergraph $\mathcal{H}(G) = (V(G), \mathcal{E})$ which has the same vertices as G and whose *hyperedges* are the *maximal cliques* of G (a *clique* is a complete induced subgraph of G, and it is *maximal* if it is not properly contained in any other clique). A k-coloring of $\mathcal{H}(G)$ is also called a *k-clique-coloring* of G, and the chromatic number $\chi(\mathcal{H}(G))$ of $\mathcal{H}(G)$ is called the *clique-chromatic number* of G, denoted by $\chi_C(G)$. A coloring of $\mathcal{H}(G)$ is *strong* if no triangle of G is monochromatic. If $\mathcal{H}(G)$ is (strong) k-colorable we say that G is (strong) *k-clique-colorable*.

Note that what we call k-clique-coloring here is also called *weak k-coloring* by Andreae, Schughart and Tuza in [1,3] or *strong k-division* by Hoáng and McDiarmid in [12]. Clearly, any (vertex) k-coloring of G is a k-clique-coloring of G, so $\chi_C(G) \leq \chi(G)$. On the other hand, note that if G is triangle-free (contains no clique on three vertices), then $\mathcal{H}(G) = G$,

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which implies $\chi_C(G) = \chi(G)$. Since the chromatic number of triangle-free graphs is known to be unbounded [18], we get that the same is true for the clique-chromatic number.

The clique-hypergraph coloring problem was posed by Duffus et al. [9]. In general, clique-coloring can be a very different problem from ordinary vertex coloring [2]. Clique-coloring is harder than ordinary vertex coloring: it is coNP-complete even to check whether a 2-clique-coloring is valid [2]. The complexity of 2-clique-colorability is investigated in [14], where they show that it is NP-hard to decide whether a perfect graph is 2-clique-colorable. However, it is not clear whether this problem belongs to NP. Recently, Marx [16] proved that it is $\sum_2^p$-complete to check whether a graph is 2-clique-colorable. On the other hand, Bacsó et al. [2] proved that almost all perfect graphs are 3-clique-colorable. Andreae et al. [1] gave a necessary and sufficient condition for $\chi_C(L(G)) \le k$ on line graph $L(G)$ of a graph G. Recently, Campos et al. [5] showed that powers of cycles is 2-clique-colorable, except for odd cycles of size at least five, that need three colors, and showed that odd-seq circulant graphs are 4-clique-colorable. Many papers focus on finding the classes of graphs G with $\chi_C(G) = 2$. Bacsó et al. [2] showed that claw-free graphs with no odd hole (which includes claw-free prefect graphs) are 2-clique-colorable. Bacsó and Tuza [3] showed that claw-free graphs of maximum degree at most four, other than an odd cycle, are 2-clique-colorable. Many subclasses of odd-hole-free graphs have been studied and shown to be 2-clique-colorable [6,7,9]. Other works considering the clique-hypergraph coloring problem in classes of graphs can be found in the literature [11–13,15].

For planar graphs, Mohar and Škrekovski [17] have shown that every planar graph is 3-clique-colorable, and Kratochvíl and Tuza [14] proposed a polynomial-time algorithm to decide if a planar graph is 2-clique-colorable (the set of cliques is given in the input). Recently, Shan, Liang and Kang [21] proved that claw-free planar graphs that are not odd cycles are 2-clique-colorable.

Theorem 1.1. *(See Mohar and Škrekovski [17].) Every planar graph is strongly 3-clique-colorable.*

Theorem 1.2. *(See Shan, Liang and Kang [21].) Every claw-free planar graph, different from an odd cycle, is 2-clique-colorable.*

The purpose of this paper is to generalize Theorem 1.1 and Theorem 1.2 to K_5-minor-free graphs and K_5-subdivision-free graphs, respectively. Section 2 gives some notation and terminology. In Section 3, we first show that every edge-maximal K_4-subdivision-free graph is 2-clique-colorable. Secondly, we show that every K_5-minor-free graph is strongly 3-clique-colorable. However, the clique-coloring problem of K_5-subdivision-free graphs remains open. In Section 4, we prove that every {claw, K_5-subdivision}-free graph G, different from an odd cycle, is 2-clique-colorable and a 2-clique-coloring can be found in polynomial time.

2. Preliminaries

Let G be an undirected simple graph with *vertex set* $V(G)$ and *edge set* $E(G)$. If H is a subgraph of G, then the vertex set of H is denoted by $V(H)$. For $v \in V(G)$, the *open neighborhood* $N(v)$ of v is $\{u: uv \in E(G)\}$, and the *closed neighborhood* $N[v]$ of v is $N(v) \cup \{v\}$. The *degree* of the vertex v, written $d_G(v)$ or simply $d(v)$, is the number of edges incident to v, that is, $d_G(v) = |N(v)|$. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G|S$. As usual, $K_{m,n}$ denotes a complete bipartite graph with classes of cardinality m and n; K_n is the complete graph on n vertices, and C_n is the cycle on n vertices. The graph $K_{1,3}$ is also called a *claw*, and K_3 a *triangle*. The graph $K_4 - e$ (obtained from K_4 by deleting one edge) is called a *diamond*. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Any graph derived from a graph F by a sequence of edge subdivisions is called a *subdivision* of F or an F*-subdivision*. A graph H is an F*-minor* if F can be obtained from H by means of a sequence of vertex and edge deletions and edge contractions, and the graph F is a *minor* of H. A graph G is F*-subdivision-free* if G has no F-subdivision as a (not necessarily induced) subgraph. A graph G is F*-minor-free* if G has no F-minor. For a family $\{F_1, \dots, F_k\}$ of graphs, we say that G is $\{F_1, \dots, F_k\}$-free if it is F_i*-free* for all i. Obviously, any graph G which contains an F-subdivision also has an F-minor. Thus an F-minor-free graph is necessarily F-subdivision-free, but not conversely. However, if F is a graph of maximum degree three or less, any graph which has an F-minor also contains an F-subdivision (see, [4, page 269]). So a graph does not contain a K_4-minor if and only if it does not contain a K_4-subdivision. The family of K_5-subdivision (-minor)-free graphs is a generalization of the planar graphs.

For an integer k, a clique of size k of a graph G is called a k*-clique* of G. The largest such k is the *clique number* of G, denoted $\omega(G)$. A subset I of vertices of G is called an *independent set* of G if no two vertices of I are adjacent in G. The maximum cardinality of an independent set of G is the *independence number* $\alpha(G)$ of G. A set $D \subseteq V(G)$ is called a *clique-transversal set* of G if D meets all cliques of G, i.e., $D \cap V(C) \neq \emptyset$ for every clique C of G. The *clique-transversal number*, denoted by $\tau_C(G)$, is the cardinality of a minimum clique-transversal set of G. The notion of clique-transversal set in graphs can be regarded as a special case of the transversal set in hypergraph theory. Erdős et al. [10] have proved that the problem of finding a minimum clique-transversal set for a graph is NP-hard. It is therefore of interest to determine bounds on the clique-transversal number of a graph. In [10] Erdős et al. proposed to find sharp estimates on the clique-transversal number $\tau_C(G)$ for particular classes of graphs G (planar graphs, perfect graphs, etc.).

Let G be a planar graph and C a cycle of G. The *interior* Int(C) of C denotes the subgraph of G consisting of C and all vertices and edges in the disk bounded by C. Similarly, Ext(C) $\subseteq G$ is the *exterior* of C. Obviously, Int(C) $\cap$ Ext(C) $= C$.