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On the complexity of the vector connectivity problem

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ABSTRACT

We study a relaxation of the Vector Domination problem called Vector Connectivity (VecCon). Given a graph G with a requirement $r(v)$ for each vertex v, VecCon asks for a minimum cardinality set S of vertices such that every vertex $v \in V \setminus S$ is connected to S via $r(v)$ disjoint paths. In the paper introducing the problem, Boros et al. [4] gave polynomial-time solutions for VecCon in trees, cographs, and split graphs, and showed that the problem can be approximated in polynomial time on n-vertex graphs to within a factor of $\log n + 2$, leaving open the question of whether the problem is NP-hard on general graphs. We show that VecCon is APX-hard in general graphs, and NP-hard in planar bipartite graphs and in planar line graphs. We also generalize the polynomial result for trees by solving the problem for block graphs.

1. Introduction and background

Recently, Boros et al. [4] introduced the Vector Connectivity problem (VecCon) in graphs. This problem takes as input a graph G and an integer $r(v) \in \{0, 1, \dots, d(v)\}$ for every vertex v of G, where $d(v)$ denotes the degree of v, and the objective is to find a vertex subset S of minimum cardinality such that every vertex v either belongs to S, or is connected to at least $r(v)$ vertices of S by vertex-disjoint paths. If we require each path to be of length exactly 1, we get the well-known Vector Domination problem [10], which is a generalization of the famous Dominating Set and Vertex Cover problems.

The Vector Connectivity problem is one of the several problems related to domination and connectivity, which have seen renewed attention in the last few years in connection with the flourishing area of information spreading (see, e.g., [10,11,7,8,1,5,6] and references therein quoted). In a viral marketing campaign one of the problems is to identify a set of targets in a (social) network that can be influenced (e.g., on the goodness of a product) and such that from them most/all the network can be influenced (e.g., convinced to buy the product). The model is based on the assumption that each vertex has a threshold $r(v)$ such that when $r(v)$ neighbors are influenced, also v will get convinced. Assume now that for v it is not enough that $r(v)$ neighbors are convinced about the product. Vertex v also requires that their motivations are independent. A way to model this "skeptic" variant of influence spreading is to require that each vertex in the network must be reached by $r(v)$ vertex-disjoint paths originating in the target set.

Another scenario where the vector connectivity problem arises is the following: Each vertex in a network produces a certain amount of a given good. We want to place in the network warehouses where the good can be stored. For secu-

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rity/resilience reasons it is better if from each source to each destination (warehouse) only a small amount of the good (e.g., one unit) travels at once. In particular, it is preferred if the units of good from one location to the different warehouses travel on different routes. This reduces the risk that if delivery gets intercepted or attacked or interrupted by a fault on the network a large amount of the good gets lost. It is not hard to see that finding the minimum number of warehouses given the amount of units produced at each vertex coincides with the vector connectivity problem.

Boros et al. developed polynomial-time algorithms for VecCon on split graphs, cographs, and trees, and showed how to model the problem as a minimum submodular cover problem, leading to a polynomial-time algorithm approximating VecCon within a factor of $\ln n + 2$ on all n-vertex graphs. One of the questions left open in that paper was whether on general graphs, VecCon is polynomially solvable or NP-hard. In this paper, we answer this question by showing that VecCon is APX-hard (and consequently NP-hard) in general graphs. Our reduction is from the Vertex Cover problem in cubic graphs. Simple modifications of the hardness proof allow us to also show that VecCon is hard in several graph classes for which the Vertex Cover problem is polynomially solvable, such as bipartite graphs and line graphs.

Our hardness results remain valid for input instances in which the vertex requirements $r(v)$ are bounded by 4. On the other hand, we show that VecCon can be solved in polynomial time for requirements bounded by 2, thus leaving open only the case with maximum requirement 3. We also develop a polynomial-time solution for VecCon in block graphs, thereby generalizing the result by Boros et al. [4] showing that VecCon is polynomial on trees. This result is obtained by introducing a more general problem called Free-Set Vector Connectivity (FreeVecCon for short), and developing an algorithm that reduces an instance of FreeVecCon to solving instances of FreeVecCon on biconnected components of the input graph.

Vertex covers and dominating sets in a graph G can be easily characterized as hitting sets of derived hypergraphs (of G itself, and of the closed neighborhood hypergraph of G, respectively). We give a similar characterization of vector connectivity sets.

The paper is structured as follows. In Section 1.1, we collect all the necessary definitions. In Section 2, we develop a characterization of vector connectivity sets as hitting sets of a derived hypergraph, which is followed with hardness results in Section 3. A polynomial reduction for a problem generalizing VecCon to biconnected graphs is given in Section 4. Some of the algorithmic consequences of this reduction are examined in Section 5. We conclude the paper with some open problems in Section 6.

1.1. Definitions and notation

All graphs in this paper are simple and undirected, and will be denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges. We use standard graph terminology. In particular, the degree of a vertex v in G is denoted by $d_G(v)$, the neighborhood and the closed neighborhood of a vertex v are denoted by $N_G(v)$ and $N_G[v]$, respectively, and $V(G)$ refers to the vertex set of G. Moreover, for a set $X \subseteq V(G)$, we define $N_G(X) = (\cup_{v\in X} N_G(v)) \setminus X$ and denote by $G[X]$ the subgraph of G induced by X. Given a graph $G = (V, E)$, a set $S \subseteq V$ and a vertex $v \in V \setminus S$, a *v–S fan of order k* is a collection of k paths $P_1, \ldots, P_k$ such that (1) every P_i is a path connecting v to a vertex of S, and (2) the paths are pairwise vertex-disjoint except at v, i.e., for all $1 \le i < j \le k$, it holds that $V(P_i) \cap V(P_j) = \{v\}$.

Given a graph $G = (V, E)$ and an integer-valued function $r : V \to \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, a *vector connectivity set* for (G, r) is a set $S \subseteq V$ such that there exists a v–S fan of order $r(v)$ for every $v \in V \setminus S$. We say that $r(v)$ is the *requirement* of vertex v. The VecCon problem is the problem of finding a vector connectivity set of minimum size for (G, r). The minimum size of a vector connectivity set for (G, r) is denoted by $\kappa(G, r)$. In Boros et al. [4], it was assumed that vertex requirements do not exceed their degrees. Since our polynomial results are developed using the more general variant of the problem, we do not impose this restriction. At the same time, our hardness results also hold for the original, more restrictive variant.

For every $v \in V$ and every set $S \subseteq V \setminus \{v\}$, we say that v is *r-linked* to S if there is a v–S fan of order r in G. Hence, given an instance (G, r) of VecCon, a set $S \subseteq V$ is a vector connectivity set for (G, r) if and only if every $v \in V \setminus S$ is $r(v)$-linked to S. Given a vertex requirement function $r : V \to \mathbb{Z}$ and a non-empty set $X \subseteq V$, we define $R(X) := \max_{x\in X} r(x)$. A graph G is *k-connected* if $|V(G)| > k$ and for every $S \subseteq V(G)$ with $|S| < k$, the graph $G - S$ is connected.

2. A characterization of vector connectivity sets

Menger's Theorem [14] implies the following characterization of vector connectivity sets, showing that they are exactly the hitting sets of a certain hypergraph derived from graph G and vertex requirement function r. This characterization will be used in our proof of Theorem 1 in Section 3.

Proposition 1. *For every graph $G = (V, E)$, vertex requirements $r : V \to \mathbb{Z}_+$, and a set $S \subseteq V$, the following conditions are equivalent:*

(i) *S is a vector connectivity set for (G, r).*
(ii) *For every non-empty set $X \subseteq V$ such that $G[X]$ is connected and $R(X) > |N_G(X)|$, we have $S \cap X \neq \emptyset$.*

Proof. First, let S be a vector connectivity set for (G, r). Suppose for a contradiction that there exists a non-empty set $X \subseteq V$ such that $G[X]$ is connected, $R(X) > |N_G(X)|$, and $S \cap X = \emptyset$. Let $C = N_G(X)$, and let $x \in X$ be a vertex such that $r(x) > |C|$. Since $S \cap X = \emptyset$, we have $x \notin S$. Moreover, the definition of C implies that in the graph $G - C$, there is no path from x to S.