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On the expansion of three-element subtraction sets

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ABSTRACT

We study the periodicity of nim-sequences for subtraction games having subtraction sets with three elements. In particular, we give solutions in several cases, and we describe how these subtraction sets can be augmented by additional numbers without changing the nim-sequence. The paper concludes with a conjecture on ultimately bipartite subtraction games.

1. Introduction

A *subtraction game* is a two-player game involving a pile of coins and a finite set S of positive integers called the *subtraction set*. The two players move alternately, subtracting some s coins such that $s \in S$. The player who makes the last move wins. Subtraction games provide classical examples of impartial combinatorial games; see [1,3,4]. They are completely understood for two-element subtraction sets. For larger subtraction sets, they are known to be all ultimately periodic [6, p. 38], but a complete solution of these games is still not known, even for three-element subtraction sets. The purpose of this paper is to report further results in this area.

Throughout this paper, the subtraction set $S = \{s_1, s_2, \ldots, s_k\}$ will be ordered $s_1 < s_2 < \cdots < s_k$. The subtraction game with subtraction set S is denoted by $\mathcal{S}$. When we need to specify the subtraction set, we will use $\mathcal{S}(S)$ or $\mathcal{S}(s_1, s_2, \ldots, s_k)$.

For each nonnegative integer n, denote $\mathcal{G}(n)$ the *Sprague–Grundy value*, or *nim-value* for short, of the single pile of size n of the subtraction game $\mathcal{S}$. The sequence $\{\mathcal{G}(n)\}_{n\geq 0}$ is called the *nim-sequence*.

Suppose for the moment that a subtraction set $\{s_1, s_2, \ldots, s_k\}$ has $d = \gcd(s_1, s_2, \ldots, s_k) > 1$. Let $s'_i = s_i/d$ for $1 \leq i \leq k$. The nim-sequence for the game $\mathcal{S}(s_1, s_2, \ldots, s_k)$ is exactly the d-plicate of that for the game $\mathcal{S}(s'_1, s'_2, \ldots, s'_k)$. That means the former can be obtained from the latter by repeating each value of the latter exactly d times [4, p. 529]. Thus, it suffices to consider subtraction sets whose members are relatively prime.

Recall that the sequence $\{\mathcal{G}(n)\}_{n\geq 0}$ is said to be *ultimately periodic* if there exist integers $p \geq 1$ and $n_0 \geq 1$ such that $\mathcal{G}(n + p) = \mathcal{G}(n)$ for all $n \geq n_0$. The smallest such numbers n_0 and p are called the *pre-period length* and *period length* respectively [1, p. 145]. If $n_0 = 0$, the sequence is said to be *purely periodic*. A purely periodic (resp. ultimately periodic) game

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is called *bipartite* (resp. *ultimately bipartite*) if $p = 2$. (Ultimately bipartite subtraction games have alternating nim-values $0, 1, 0, 1, \dots$.) The periodicity of subtraction games is discussed in [1–4]. Bipartite games are first introduced in [5].

Throughout this paper, when saying that a subtraction game has periodic nim-values $g_1 g_2 \dots g_p$, we mean that ultimately, the nim-sequence is the infinite repetition of the subsequence $g_1, g_2, \dots, g_p$. The n_0 nim-values $\mathcal{G}(0), \mathcal{G}(1), \dots, \mathcal{G}(n_0 - 1)$ are called the pre-periodic nim-values. The omission of commas between the nim-values does not lead to any misunderstanding as all nim-values presented in this paper have exactly one digit.

The following lemma gives us a useful idea of how much calculation we need to perform to determine the pre-period and period lengths.

Lemma 1. *(See [1, p. 148].) Let $s_k = \max(S)$. For minimal n_0 and p such that $\mathcal{G}(n + p) = \mathcal{G}(n)$ for $n_0 \le n < n_0 + s_k$, the subtraction game $\mathcal{S}$ is purely periodic with the pre-period length n_0 and the period length p.*

Remark 1. (See [3, p. 84].) For a given subtraction set S, if there exists a positive integer s such that $\mathcal{G}(n + s) \ne \mathcal{G}(n)$ for all nonnegative integers n, then s can be adjoined to the subtraction set S without changing the nim-sequence. In this case, for brevity, we will simply say that *s can be adjoined to S*. The set of all such elements, including elements in S, is called the *expansion set* of S and denoted by S^{ex}.

In reality, to identify such an s, we need to calculate only a small range of nim-values, as much as the calculation needed in Lemma 1. We detail this range as follow.

Theorem 1. *Let $\mathcal{S}$ be a subtraction game with pre-period length n_0 and period length p.*

(i) *A number $s < n_0 + p$ can be adjoined to S if and only if $\mathcal{G}(n + s) \ne \mathcal{G}(n)$ for all n such that $0 \le n < n_0 + p$.*
(ii) *A number $s \ge n_0 + p$ can be adjoined to S if and only if $s - p$ can be.*

Proof. The theorem follows immediately from Remark 1 and the definition of ultimately periodicity. □

It follows from Theorem 1 that if $s \ge n_0 + p$ and s can be adjoined to S then $s = s' + mp$ for some m and s' such that $n_0 \le s' < n_0 + p$ and s' can be adjoined to S. We are going to employ this idea to represent S^{ex} with no more than $\max(n_0 + p, s_k)$ elements with $s_k = \max(S)$. (There is another method to present S^{ex} with no more than $n_0 + p$ elements but we avoid its complexity.)

Let us partition S into two parts: S_1 containing elements s of S such that $s < n_0$ and $S_2 = S \setminus S_1$. Let S'_1 (resp. S'_2) be the set of all s which can be adjoined to S such that $s \notin S$ and $s < n_0$ (resp. $n_0 \le s < n_0 + p$). Then

$$S^{ex} = \{S_1 \cup S'_1\} \cup \{S_2 \cup S'_2\}^{*p}$$

in which $T^{*p} = \{t + mp | t \in T, m \ge 0\}$. Note that in the formula for S^{ex}, some of the sets S_1, S'_1, S_2, and S'_2 may be empty. In particular, $S_1 \cup S'_1 = \emptyset$ if $\mathcal{S}$ is purely periodic ($n_0 = 0$).

Example 1. The subtraction game $\mathcal{S}(1, 8, 11, 27)$ is ultimately periodic with $n_0 = 13$, $p = 19$ and has expansion set $S^{ex} = \{1, 8, 11\} \cup \{13, 20, 27\}^{*19}$. Here $S_1 = \{1, 8, 11\}$, $S'_1 = \emptyset$, $S_2 = \{27\}$, and $S'_2 = \{13, 20\}$.

Definition 1. If $S^{ex} = \{S_1\} \cup \{S_2\}^{*p}$ (equivalently, $S'_1 \cup S'_2 = \emptyset$), the set S is said to be *non-expandable*. Otherwise, it is *expandable*.

Some values of the expansion sets of certain subtraction sets with numbers up to 7 can be found in [3, pp. 84–85].
The following result from [5] shows that for bipartite games, S^{ex} is the set of odd positive integers.

Theorem 2. *Let S be a subtraction set with $\gcd(S) = 1$. The subtraction game $\mathcal{S}$ is bipartite if and only if $1 \in S$ and the elements of S are all odd.*

Remark 2. The key results of this paper are the determinations of periodic nim-values and expansion sets in several specific cases. We do not give the proofs for all results. There are considerable similarities in the proofs of various results. We give one proof for the expansion set of the subtraction set $\{a, b\}$ and one proof for the nim-sequence for the subtraction game $\mathcal{S}(1, a, b)$. The reason is simply that these two games are addressed early in the paper. The details of proofs for other stated results are similar; some are quite tedious but they are all straightforward.

This paper is organized as follows. Section 2 lists our results on 5 classes of subtraction sets: $\{a, b\}$, $\{1, a, b\}$, $\{a, b, a + b\}$, $\{a, b, a + b + 1\}$, and $\{a, b, a + b + 2ja\}$. We consider various cases for each class. Section 3 gives two proofs as mentioned in Remark 2. We provide proofs in Section 3 rather than in Section 2 so that the reader can follow Section 2 easily. In Section 4,