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Multi-Path Algorithms for minimum-colour path problems with applications to approximating barrier resilience

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A B S T R A C T

Let G be a graph with zero or more colours assigned to its vertices, and let v_s and v_t be two vertices of G. The *minimum-colour path problem* is to determine the minimum over all v_s–v_t paths of the number of colours used, where a colour is considered used if it is assigned to any vertex in the path. Although this problem is NP-hard with strong hardness of approximation results, many problems can be formulated as instances of the minimum-colour path problem with additional constraints which may be exploited to allow polynomial-time solutions or close approximations. We introduce a family of approximation algorithms, referred to as the *Multi-Path Algorithms*, for minimum-colour path problems, and go on to show examples of constraints which would allow polynomial-time solutions or constant factor approximations.

In particular, we describe applications to variants of the *barrier resilience problem*: given a pair of points s and t and an arrangement $\mathcal{A}$ of n regions in the plane, the problem is to determine the minimum over all s–t paths of the number of regions intersected. We show how to reduce the barrier resilience problem to the minimum-colour path problem, and go on to show that the Multi-Path Algorithms guarantee a 1.5 approximation when regions are unit disks and s, t are separated by at least $2\sqrt{3}$.

1. Introduction

Let the *minimum-colour path problem* be defined as follows:

Minimum-Colour Path Problem (MCPP)
Input: $(G = (V, E, C, \mathcal{C}), v_s, v_t)$, where:

- G is a vertex-coloured graph comprising the set of vertices V, the set of directed edges E, the set of colours C, and the colour assignment function $\mathcal{C} : V \to 2^C$, such that for every vertex v in V, $\mathcal{C}(v)$ denotes the set of colours assigned to v.
- v_s and v_t are a pair of uncoloured vertices in V.

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Given a path P on G, let $\mathcal{C}(P)$ denote the set of colours used by P, defined as the union over all sets of colours assigned to vertices in P, including the vertices at the ends of P. Then let $\chi(P)$ denote the *chromatic length* of P, defined as $|\mathcal{C}(P)|$. Finally, for every pair of vertices v and w, let $\chi(v, w)$ denote the minimum, over all paths P from v to w, of $\chi(P)$.
Output: $\chi(v_s, v_t)$

We note that some of the works in the literature use different variants of the MCPP where colours can be assigned to edges and G is allowed to be a multi-graph. However, it is straightforward to perform polynomial-time approximation-preserving reductions from such variants by replacing every edge with a vertex of degree 2.

The MCPP has attracted the interest of network theorists, since many problems involving the reliability of a network can be formulated as MCPP instances [1–5]. Furthermore, given the amount of freedom in assigning colours, several other problems can also be reduced to the MCPP [1,5,6], leading to applications in other areas such as barrier coverage [7].

However, such problems also include NP-complete problems, thus many variants of the MCPP, including the version defined above, have been shown to be NP-complete [1,5,6]. Furthermore, because of a reduction from the red–blue set cover problem, unless there exist quasi-polynomial-time algorithms for solving NP-complete problems, there also cannot exist a polynomial-time $O(2^{(\log|C|)^{1-\epsilon}})$, $O(2^{(\log|V|)^{1-\epsilon}})$, or $O(2^{(\log|E|)^{1-\epsilon}})$-approximation algorithm for the MCPP for any $\epsilon > 0$ [5,6].

Nevertheless, because of the problem's significance, many researchers have turned to analyzing heuristic algorithms in hopes of finding algorithms which work well in practice [1,2], or analyzing restricted variants of the problem in hopes of identifying large classes of problem instances which can be solved or closely approximated within polynomial time [3–5,8].

An example of such restricted variants is the case where the vertices of G form a path, cycle or a tree, but G is allowed to be a multi-graph with colours assigned to edges, thus a path connecting a specified pair of vertices may not be unique. For this special case, there exists a polynomial-time $O(\log|V|)$-approximation algorithm [5]. However, unless there exist quasi-polynomial-time algorithms for solving NP-complete problems, there cannot exist a polynomial-time $(1-\epsilon)(\ln|C|)$-approximation algorithm for any $\epsilon > 0$ [5,9].

In this paper, we develop a family of approximation algorithms referred to as the Multi-Path Algorithms for the MCPP. In contrast to the works cited above which focus on analyzing versions of the MCPP where additional constraints are placed on the structure of the entire graph, the Multi-Path Algorithms are designed to work well when the additional constraints guarantee the existence of a v_s–v_t path with added structural constraints. This approach is motivated by computational geometry problems, where there are often proofs that show the existence of some optimal path with complex structural constraints.

In particular, we note that the Multi-Path Algorithms are effective in finding near optimal solutions to MCPP instances derived from the *barrier resilience problem* in the field of sensor networks, defined as follows:

Barrier Resilience Problem (BRP)
Input: $(\mathcal{A}, s, t)$, where:

- $\mathcal{A}$ is an arrangement of n regions in a two-dimensional plane, where each region represents the detection region of a sensor.
- s and t are a pair of points in the two-dimensional plane which are not covered by any of the sensor detection regions in $\mathcal{A}$.

Output: The s, t*-resilience* of $\mathcal{A}$ with respect to s and t, denoted $\rho(\mathcal{A}, s, t)$, defined as the minimum over all s–t paths of the number of distinct sensor detection regions intersected.

In previous work, Bereg and Kirkpatrick [7] described a polynomial-time approximation-preserving reduction from the BRP to an edge-coloured variant of the MCPP. In Section 5, we describe the BRP in more detail and show that it can also be reduced to the variant of the MCPP defined in this paper. Furthermore, to illustrate one of the applications for the Multi-Path Algorithms, we will analyze the following restricted variant of the BRP proposed by Bereg and Kirkpatrick [7]:

Barrier Resilience Problem – Unit Disks, Moderate Separation (BRP-UDMS)
Input: $(\mathcal{A}, s, t)$, where:

- $\mathcal{A}$ is an arrangement of n unit disk regions in a two-dimensional plane, where each region represents the detection region of a sensor.
- s and t are a pair of *moderately separated* (separated by at least $2\sqrt{3}$) points in the two-dimensional plane which are not covered by any of the sensor detection regions in $\mathcal{A}$.

Output: The s, t-resilience of $\mathcal{A}$ with respect to s and t, denoted $\rho(\mathcal{A}, s, t)$, defined as the minimum over all s–t paths of the number of distinct sensor detection regions intersected.