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A fixed point theorem for non-monotonic functions ☆

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ABSTRACT

We present a fixed point theorem for a class of (potentially) non-monotonic functions over specially structured complete lattices. The theorem has as a special case the Knaster–Tarski fixed point theorem when restricted to the case of monotonic functions and Kleene's theorem when the functions are additionally continuous. From the practical side, the theorem has direct applications in the semantics of negation in logic programming. In particular, it leads to a more direct and elegant proof of the least fixed point result of [12]. Moreover, the theorem appears to have potential for possible applications outside the logic programming domain.

1. Introduction

The problem of *negation-as-failure* [1,9] in logic programming has received considerable attention for more than three decades. This research area has proven to be a quite fruitful one, offering results that range from the very practical to the very theoretical. On the practical side, negation-as-failure is nowadays used in various areas of Computer Science (such as in databases, artificial intelligence, and so on). On the more theoretical side, the study of negation-as-failure has triggered the deeper study of the nature and repercussions of non-monotonicity in Computer Science. In particular, the study of the *meaning* of logic programs with negation has made evident the necessity of a fixed point theory for non-monotonic functions.

The fixed point semantics of classical logic programs (i.e., programs without negation in the bodies of rules) was developed by van Emden and Kowalski [15] and is based on classical fixed point theory (in particular on the least fixed point theorem of Kleene). However, if negation is introduced in logic programs, the traditional tools of fixed point theory are no longer applicable due to the non-monotonicity of the resulting formalism. A crucial step in the study of logic programs with negation was the introduction of the *well-founded semantics* [17] which employs a three-valued logic in order to capture the meaning of these programs. It has been demonstrated that every such program possesses a minimal three-valued model which can be constructed as the least fixed point (with respect to the so-called Fitting ordering [9]) of an appropriate operator associated with the program. The well-founded approach triggered an increased interest in the study of non-monotonic functions. Such a study aims at developing an abstract fixed point theory of non-monotonicity which will

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have diverse applications in various disciplines and research areas. Results in this direction have been reported in [6,7,18]. A detailed account of these results and their relationship with the work developed in this paper will be given in Section 8. As a general statement we can say that the existing results indicate that non-monotonic fixed point theory is a deep area of research that certainly deserves further investigation.

The purpose of the present paper is to develop a novel fixed point theory for an interesting class of non-monotonic functions. Our motivation comes again from the area of logic programming with negation. However, our starting point is not one of the traditional constructions of the well-founded semantics (such as for example [11,16]). Instead, we start from the *infinite-valued semantics* [12] which is a relatively recent construction that was developed in order to give logical justification to the well-founded approach. In the infinite-valued semantics the meaning of logic programs with negation is expressed through the use of an infinite-valued logic. Actually, as it is demonstrated in [12], every logic program with negation has a unique *minimum infinite-valued* model which is the *least fixed point* of an operator with respect to an ordering relation. This minimum model result extends the well-known minimum model theorem that holds for definite logic programs. Moreover, the infinite-valued construction is compatible with the well-founded semantics since, as shown in [12], if we restrict the minimum infinite-valued model to a three-valued logic, we get the well-founded model. It is therefore natural to wonder whether the infinite-valued semantics can form the basis for a novel fixed point theory of non-monotonicity.

In order to develop such a fixed point theory, we keep the essence of the set-theoretic constructions of [12] but abstract away from all the logic programming related issues. In particular, instead of studying the set of interpretations of logic programs we consider abstract sets that possess specific properties. Moreover, instead of focusing on functions from interpretations to interpretations, we consider functions from abstract sets to abstract sets. More specifically, our starting point is a complete lattice $(L, \leq)$ equipped with a family of preorderings indexed by ordinals that give rise to an ordering relation $\sqsubseteq$. We demonstrate that if the preorderings over L obey certain simple and natural axioms, then the structure $(L, \sqsubseteq)$ is also a complete lattice. We then prove that a large class of functions $f : L \to L$ which may not be monotonic with respect to $\sqsubseteq$, possess a least fixed point with respect to $\sqsubseteq$. Moreover, we demonstrate that our theorem generalizes both the Knaster–Tarski and the Kleene fixed point theorems (when f is monotonic or continuous respectively).

The contributions of the present work can be summarized as follows:

- We develop a fixed point theorem for a class of (potentially) non-monotonic functions over specially structured complete lattices. The structure of our lattices stems from a simple set of axioms that the corresponding ordering relations have to obey. The proposed fixed point theorem appears to be quite general, since, apart from being applicable to a large class of non-monotonic functions, it also generalizes well-known fixed point theorems for monotonic functions.
- We demonstrate the versatility of the proposed theorem by deriving a much shorter and cleaner proof of the main theorem of [12]. Actually, we demonstrate a much stronger result which may be applicable to richer extensions of logic programming.
- We argue that the proposed theorem may be applicable to other areas apart from logic programming. In particular, we demonstrate that the axioms on which the fixed point theorem is based, have a variety of other models apart from the set of interpretations of logic programs. This fact additionally advocates the generality of the proposed approach.

The rest of the paper is organized as follows: Section 2 introduces the infinite-valued approach which motivated the present work. This introduction to the infinite-valued approach facilitates the understanding of the more abstract material of the subsequent sections; moreover, as we are going to see, the infinite-valued approach will eventually benefit from the abstract setting that will be developed in the paper. Section 3 introduces the partially ordered sets that will be the objects of our study. Every such set is equipped with an ordering relation whose construction obeys four simple axioms. The main properties of these sets are investigated. In Section 4 it is demonstrated that every partially ordered set whose ordering relation satisfies the axioms of Section 3, is a complete lattice. Section 5 presents certain complete lattices that satisfy the proposed axioms. Section 6 develops the novel fixed point theorem for functions defined over the specially structured complete lattices introduced in the preceding sections. Section 7 demonstrates a large class of functions over which the new fixed point theorem is applicable. As it turns out, the immediate consequence operator for logic programs with negation falls into this class. In this way we obtain the main result of [12] as a special case of a much more general theorem. Section 8 provides a comparison with related work and Section 9 concludes the paper with pointers to future work.

In the following, we assume familiarity with the basic notions regarding logic programming (such as for example [10]) and of partially ordered sets and particularly lattices (such as for example [5]).

2. An overview of the infinite-valued approach

In this section we provide the basic notions and definitions behind the infinite-valued approach; our presentation mostly follows [12]. Some standard technical terminology regarding logic programming (such as “atoms”, “literals”, “head/body of a rule”, “ground instance”, “Herbrand Base”, and so on), will be used without further explanations (see [10] for a basic introduction).

Definition 2.1. A (first-order) *normal program rule* is a rule with an atom as head and a conjunction of literals as body. A (first-order) *normal logic program* is a finite set of normal program rules.