Theoretical Computer Science 546 (2014) 17–29

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Cartesian closed categories of separable Scott domains

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ARTICLE INFO

Keywords:
Scott domain
Cartesian closed category
Lambda calculus

ABSTRACT

We classify all sub-cartesian closed categories of the category of separable Scott domains. The classification employs a notion of coherence degree determined by the possible inconsistency patterns of sets of finite elements of a domain. Using the classification, we determine all sub-cartesian closed categories of the category of separable Scott domains that contain a universal object. The separable Scott domain models of the $\lambda\beta$-calculus are then classified up to a retraction by their coherence degrees.

1. Introduction

We revisit some classical themes in domain theory: models of the λ-calculus, universal domains, and cartesian closed categories (ccc's) of domains. In [1], Scott showed that $\mathcal{P}(\omega)$, the partial order of all sets of natural numbers, is universal in the cartesian closed category of separable continuous lattices and continuous functions, by which is meant that $\mathcal{P}(\omega)$ is in this category and every object in the category is a retract of it. As it is universal in a cartesian closed category, $\mathcal{P}(\omega)$ is necessarily a model of the $\lambda\beta$-calculus because its function space is then a retract of it.

Next, in [2], Plotkin introduced the cartesian closed category of coherent separable continuous bounded complete pointed domains, and showed that it contains a universal object $\mathbb{T}^{\omega}_{\perp}$.

Then, in [3], Scott introduced the category of Scott domains, and gave a universal object for the full sub-cartesian closed category of the separable Scott domains. In [4] he gave another universal object for this category: the partial order of all consistent propositional theories over a given countably infinite set of propositional letters. It is worth noting that in [1], Scott had already essentially considered the separable Scott domains in terms of closed subsets of retracts of $\mathcal{P}(\omega)$.

With the introduction of powerdomains [5,6], attention was also paid to wider categories of separable algebraic domains, and eventually interest arose in finding characterisations of cartesian closed categories by maximality properties. The first of these was given by Smyth [7]; it characterises the separable bifinite domains as the largest full subcategory of the separable pointed algebraic domains that is a sub-ccc of the category of all directed complete partial orders (dcpo's). Later, Jung introduced his FS domains, and gave a characterisation [8,9] of them as the largest full subcategory of the separable pointed (by definition continuous) domains that is a sub-ccc of the category of all pointed dcpo's.

We consider three natural classification questions concerning the category **Dom** of separable Scott domains. The first asks which such Scott domains are models of the $\lambda\beta$-calculus, up to retraction; the second asks which retract closed full

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http://dx.doi.org/10.1016/j.tcs.2014.02.042
0304-3975/

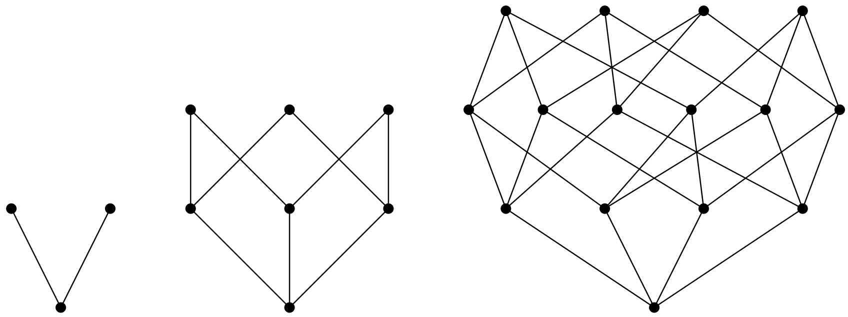

Fig. 1. The first three truncated hypercubes $\mathbb{T}_2$, $\mathbb{T}_3$, and $\mathbb{T}_4$.

sub-cartesian closed categories of **Dom** have universal objects; and the third asks, more broadly, for the classification of all retract closed full sub-cartesian closed categories of **Dom**. We answer all three questions. The answers to the first two are straightforward once the classification is available (although not all the classification is needed).

Regarding the third question, there is an obvious remark: as well as the three categories already mentioned, there are also their three full subcategories of finite domains. Less obviously, the notion of coherence generalises: say that a Scott domain is *n-coherent* when any subset of size at least n is bounded if all its subsets of size n are bounded; the ω-algebraic lattices arise when $n = 0$ and when $n = 1$, and the coherent domains arise when $n = 2$. For $n \geqslant 2$, define the *truncated n-dimensional hypercube* $\mathbb{T}_n$ to be the n-dimensional hypercube ordered outwards from the origin and with the maximal point removed. Fig. 1 depicts the first three truncated hypercubes. Then $\mathbb{T}_n$ is a simple example of a domain that is n-coherent, but not $(n-1)$-coherent. Note that $\mathbb{T}_2$ is $\mathbb{T}_\perp$.

It is not hard to show that the full subcategories of n-coherent separable domains provide further examples of retract closed full sub-ccc's of **Dom**, and so does their union, and all the full subcategories of domains with finitely many points. However this does not exhaust all the possibilities, as points can participate in truncated hypercubes of all dimensions. So we can classify points according to the pattern of their participation in truncated hypercubes and, in turn, we can classify other points according to their participation in such participations, and so on. This leads to well-founded countably branching trees of points and so to countable ordinals. These, in turn allow us to assign ordinal-valued invariants called *coherence degrees*, first to domains, and then to categories of them. Armed with these invariants and two other rather simpler ones (whether a domain is finite and the cardinality of its set of maximal points), one can give a complete classification of the retract closed full sub-ccc's of **Dom**.

After giving preliminary definitions and notation for domain theory in Section 2, we define coherence degrees of domains and categories in Section 3. Next, in Section 4, we calculate coherence degrees; this enables us to prove Theorem 4.9 showing that categories defined in terms of their coherence degrees are sub-ccc's of **Dom**. In Section 5 we analyse domains in terms of retracts of coherent powers of standard domains determined by the coherence degrees of the domains under analysis. This enables us to prove Theorem 5.6 showing that retract closed full sub-ccc's of **Dom** containing the domain of natural numbers can be characterised in terms of their coherence degrees. Section 6 considers the remaining class of categories, those whose objects have finitely many maximal points. Putting all this together in Section 7, we prove our classification theorem, Theorem 7.1. We are then able to show which of the categories classified have universal domains, see Theorem 7.3, and to identify all retract closed full sub-ccc's of **Dom** generated by a model of the $\lambda\beta$-calculus, see Theorem 7.4. Finally, Theorem 7.6 gives a classification of all locally-monotone fully faithful functors between the ccc's given by the classification theorem.

Regarding future work, a question of immediate interest is extending our results to the, so-to-speak, continuous Scott domains, more precisely, to the separable consistently-complete pointed domains. Regarding algebraic domains, having Smyth's result on the maximal ccc of separable pointed algebraic domains, one can ask for a classification of all such categories; with Jung's result, there is an analogous question for the separable domains. In a different direction, having available a classification of all separable Scott domain models of the $\lambda\beta$-calculus, one could investigate the relations between the corresponding realisability models.

2. Domain theory

We generally use the terminology of [10]. In particular, *Scott domains* are the algebraic bounded complete pointed domains. The *separable* algebraic domains are those with countably many finite elements, and we write D^{o} for the set of finite elements of an algebraic domain D. A subset of a boundedly complete partial order is *consistent* if it has an upper bound, equivalently if it has a least upper bound. A *minimally inconsistent (mic)* subset of a bounded complete partial order is one that is inconsistent, but all of whose proper subsets are consistent; every such set is finite. If $\{x_0, \ldots, x_{n-1}\}$ is a mic set in a Scott domain then there are finite $a_i \leqslant x_i$, such that $\{a_0, \ldots, a_{n-1}\}$ is also a mic set.

A *basis* of a Scott domain is a set of finite elements of the domain such that every finite element is a least upper bound of elements of the basis. The cartesian product $D \times E$ of two Scott domains is a Scott domain and its finite elements have