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Sobriety for equilogical spaces ☆

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ABSTRACT

The category of equilogical spaces, as well as the exact completions of the category of T_0-spaces and of the category of topological spaces, offers locally cartesian closed extensions of the category of topological spaces. Hence in any one of such categories, it is straightforward to consider spaces of continuous functions without bothering about ensuring that they be topological spaces.

We test this fact with the notion of sober topological space, producing a synthetic characterization of those topological spaces which are sober in terms of a construction on equilogical spaces of functions.

1. Introduction

Natural categories in modeling computation consist of topological spaces obtained from directed-complete partial orders endowed with the topology of directed sups (often called the Scott topology). These are cartesian closed, but need not be closed under various useful constructions like subspaces or quotients which are otherwise natural in the approach using (one of the various notions of) filter spaces. In an attempt to reunify the views, Dana Scott proposed the category ***Equ*** of equilogical spaces: An *equilogical space* is a triple $E = (S_E, \tau_E, \equiv_E)$ where (S_E, τ_E) is a topological T_0-space and $\equiv_E \subseteq S_E \times S_E$ is an equivalence relation. A map $f : E \to E'$ between equilogical spaces is a function between the quotient sets $f : S_E/{\equiv_E} \to S_{E'}/{\equiv_{E'}}$ which has a *continuous* choice function $g : S_E \to S_{E'}$ tracking it on the representatives. In other words, for some appropriate continuous function g

$$f\left([x]_{\equiv_E}\right) = \left[g(x)\right]_{\equiv_{E'}}, \quad \text{all } x \in S_E.$$

In [1], Dana Scott had already noticed that these data form a cartesian closed category, and that proposal was reaffirmed in [2].

The construction has been compared to that of the category ***Mod*** of the modest sets in the effective topos, see [3,4]; in fact, in both situations, one can explain local cartesian closure based on the same general facts, see [5,6]. And the crucial construction involved is that of the exact completion of a category with finite limits. In that perspective, a thorough comparison between various cartesian closed extensions of the category of topological spaces was conducted in [7].

In Section 2 we recall the basic constructions of the various categories starting from that of topological spaces and explaining the underlying intuition. In Section 3 we recall a synthetic description of sobriety, as proposed in [8,7] and also

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considered for possibly different purposes in [9–11] and state the characterization theorem of sober spaces as topological spaces satisfying an intrinsic property in ***Equ***. In Section 4 we produce the proof of the characterization theorem.

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2. Viewing topological spaces as abstract sets

The intuition about expanding the notion of set as a collection of points *without any further structure* to include objects where some kind of cohesion among the points is described quite vividly in [12–14]. And the first approximation to such an intuition that one may take is to consider topological spaces and continuous functions and the category ***Top*** that they form.

One immediately notices that, in order to consider topological spaces (and continuous functions) *as sets*, it is crucial to have the construction of quotient by an equivalence relation. And the definition of exact category singles out precisely the properties of the standard construction of a set of equivalence classes, see [15]: given an equivalence relation $\sim$ on an object S, the quotient $S/{\sim}$ is the smallest solution to the problem of defining transformations which preserve the identification on arguments induced by the equivalence relation. The property (that one then checks in proving the factorization theorem for set-functions) that

$$[x]_\sim = [x']_\sim \iff x \sim x'$$

can be restated category-theoretically as saying that the kernel equivalence relation induced by the canonical surjection $S \twoheadrightarrow S/{\sim}$ coincides with the given equivalence relation $\sim$. This makes a quotient of sets *effective*. Finally, it is a property of the logic that gives *stability*: any renaming of the equivalence classes $g : X \to S/{\sim}$ is (in bijection with) the classes for an equivalence relation on $\{(x, s) \mid g(x) = [s]_\sim\}$.

Formally (and in a nutshell), given a category $\boldsymbol{C}$, an *equivalence relation in* $\boldsymbol{C}$ is a pair of arrows $A_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} A_0$ which is jointly monic, reflexive, symmetric, and transitive. A *quotient* of such an equivalence relation is a coequalizer $f : A_0 \longrightarrow A$ of the parallel pair. And the quotient is said to be *effective* if its kernel pair is isomorphic to the given equivalence relation. Moreover it is *stable* if any pullback of it is a quotient.

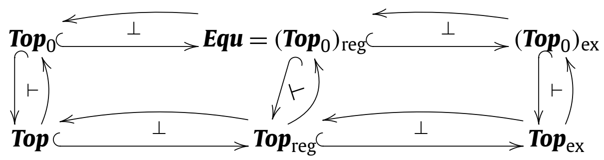
A category $\boldsymbol{C}$ is *exact* if it has finite limits, quotients of equivalence relations, and these are effective and stable under pullbacks. We refer the interested reader to [15–17,7] for a thorough discussion of the notions involved and examples.

A weaker notion is that of a *regular* category which is a category with finite limits and with quotients of kernel pairs which are effective and stable.

The category of topological spaces and continuous functions has quotients of equivalence relations, but these are *not* in general effective or stable. In other words, those quotients are *wrong* if one wants to think in terms of "spaces as sets".

Again, category theory offers a solution for free: since the notion of exact category is algebraic over categories with finite limits, there is always the free exact category generated by a category with finite limits. Moreover, as presented in a well-known paper [18], such a free construction can be obtained in finitary terms over the given category with finite limits. In the case of the free exact category $\boldsymbol{Top_0}_{\mathrm{ex}}$ over the category of T_0-spaces, one obtains an exact category which is also locally cartesian closed and extensive (in other words, a locally cartesian closed pretopos), extending the original result of [1,2], see [7].

Relating the notion of exact category to that of regular category in the sense of [15], Aurelio Carboni also produced a finitary presentation of the free regular category over a category with finite limits in [16]. And all this provides a square of product-preserving reflections



The categories in the last column are locally cartesian closed pretoposes, those in the middle column are locally cartesian closed quasitoposes, see [19]. And the inclusions preserve finite limits and all existing exponentials.

The characterization which follows can be considered in any of the completions in the diagram above. But, since we are interested in characterizing topological spaces which are T_0-spaces using a condition on exponentials, we can restrict the computation to ***Equ***, the smallest of the cartesian closed categories in the diagram above, but the final result can be extended to all of the other cartesian closed categories in the diagram above.

3. A characterization of sobriety

Let $\Sigma = (\{0, 1\}, \varsigma)$ be the Sierpinski topological space with two points: one open, one closed. The space Σ is actually an algebraic lattice endowed with its Scott topology, see Section 4 for the general definition, but recall that an algebraic lattice is a complete lattice where every element is the sup of the compact elements below it.