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Elements of a theory of algebraic theories

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ABSTRACT

Kleisli bicategories are a natural environment in which the combinatorics involved in various notions of algebraic theory can be handled in a uniform way. The setting allows a clear account of comparisons between such notions. Algebraic theories, symmetric operads and nonsymmetric operads are treated as examples.

1. Introduction

This paper has its genesis in Glynn Winskel's use of presheaf categories and profunctors in the foundations of concurrency. His basic theory is laid out in [4] with Cattani, and particular cases of Kleisli bicategories appear there. A preordered set version, providing a model for linear logic, is already in [24]. Kleisli bicategories are both a rich source of models and a context in which to understand subtle theory. Their value was recognised by a group of us in Cambridge and we set about preparing an exposition [8] of the general theory. Around the same time John Power realised the significance of the key pseudo-distributivities in connection with extensions of Edinburgh work [9] on variable binding. The paper [6] shows the common interest and in Edinburgh a thesis [22] and papers (for example [23]) quickly followed. By contrast the Cambridge exposition remains unfinished, and there is just one paper [7] which gives some sense of our preferred approach. That is my fault and I have written this paper for Glynn Winskel by way of apology. It is not intended as a substitute for the unfinished paper. Rather, it sketches applications to algebraic theories and operads, which I have presented in talks over the years. In developing the ideas, I have profited from discussions on with Richard Garner and John Power. Recently Garner and I have made progress on coalgebraic aspects, and a substantial theory is emerging. Here I focus on just one strand of ideas, and leave details and the wider perspective for other occasions.

The paper is organised as follows. In Sections 2, I describe and give elementary properties of the basic construction, that of the Kleisli bicategory $\mathrm{Kl}(P)$ of a Kleisli structure P. This is in my view a good way to understand the bicategories with which we shall be concerned, and I explain how even the basic bicategory **Prof** of profunctors, which underlies the whole paper, can be considered from this perspective. Section 3 is concerned with distributive laws, composite Kleisli structures and features of the corresponding Kleisli categories. The idea of a pseudo-distributive law at roughly the level we need is old, see [14]. Perhaps because there were no compelling applications, the details were not worked through. The first complete account seems to be [19]. The use here of Kleisli structures creates a new focus but there are no great surprises.

In Section 4, I describe my approach to algebraic theories as monads in Kleisli bicategories. A concrete categorical treatment of essentially the same point of view is in [5]. The value of an abstract treatment becomes more apparent with very recent work but even at the level of this paper I hope readers will appreciate the smooth treatment of categories generated by theories. In the final Section 5, I use the general setting to give a treatment of comparisons between notions of algebraic theory. I hope inter alia to encourage sensitivity to some subtleties in the notions of symmetric and non-symmetric operads.

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I make a small remark about notation. I have decided to use P both when describing a general Kleisli structure and for the presheaf Kleisli structure. I hope to avoid confusion by the following convention. In the general case I use lower case letter for the objects of a bicategory. In the special case when the objects are themselves categories I use upper case letters.

These introductory remarks make clear that I have discussed material with many people. But on this occasion I thank in particular Glynn Winskel. We go back a long way and it has been continually stimulating to discuss logic and computer science with him over many years. I very much appreciate his intellectual honesty and openness, and his talent for grounding abstract mathematics in the modelling of computational phenomena. This paper derives from work of his and he was the first person with whom I discussed Kleisli bicategories. I dedicate the paper to him with affection and best wishes for the future.

2. Kleisli bicategories

The Kleisli formulation of a monad on a category $\mathbb{C}$, given in [18], has not played a prominent role in mathematics, but it is familiar in the programming language community from the computational λ-calculus [20] and premonoidal categories [21]. The Kleisli presentation gives for each object c in a category $\mathbb{C}$ a unit $c \to Tc$ in $\mathbb{C}$ and for each $f : c \to Td$ in $\mathbb{C}$ a lift $f^\sharp : Tc \to Td$. This data is required to satisfy evident equations. The virtue of the formulation is that it makes trivial the definition and basic properties of the Kleisli category of a monad. Now one can still define a Kleisli category when structure is only given for some subcollection of the objects of $\mathbb{C}$ and generalisations of this kind have been identified, for example in [1]. The phrase relative monad is in use, but for the cases considered here I prefer to say restricted.

2.1. Kleisli structures

A Kleisli structure is a 2-dimensional version of a restricted monad. The starting point is a bicategory $\mathcal{K}$ equipped with a sub-bicategory $\mathcal{A} \hookrightarrow \mathcal{K}$.

Definition 2.1. A *Kleisli structure* P on $\mathcal{A} \hookrightarrow \mathcal{K}$ is the following.

- A choice for each object $a \in \mathcal{A}$, of an arrow $y_a : a \to Pa$ in $\mathcal{K}$.
- For each pair $a, b \in \mathcal{A}$ of objects, a functor
$$\mathcal{K}(a, Pb) \longrightarrow \mathcal{K}(Pa, Pb) \qquad f \longmapsto f^\sharp$$
- Families of invertible 2-cells
$$\eta_f : f \to f^\sharp.y_a \qquad \kappa_a : (y_a)^\sharp \to 1_{Pa} \qquad \kappa_{g,f} : \left(g^\sharp.f\right)^\sharp \to g^\sharp.f^\sharp$$
natural in $f : a \to Pb$ and $g : b \to Pc$ as appropriate, and subject to unit and pentagon coherence conditions.

It is clear from the data that P can be given the structure of a pseudo-functor $P : \mathcal{A} \to \mathcal{K}$. For $f : a \to b$ in $\mathcal{A}$, set $Pf = (y_b f)^\sharp : Pa \to Pb$. The 2-cell structure and its coherence are routine. Then $y_a : a \to Pa$ can be given the structure of a transformation. For $a \xrightarrow{f} b$ the structure 2-cells are $y_b.f \xrightarrow{\eta_{y_b.f}} (y_b.f)^\sharp.y_a = Pf.y_a$.

2.2. The Kleisli bicategory

Given a Kleisli structure P on $\mathcal{A} \hookrightarrow \mathcal{K}$ we define its *Kleisli bicategory* $\mathrm{Kl}(P)$ as follows. The objects of $\mathrm{Kl}(P)$ are the objects of $\mathcal{A}$. For objects a, b, set $\mathrm{Kl}(P)(a, b) = \mathcal{K}(a, Pb)$. The identities of $\mathrm{Kl}(P)$ are the $y_a : a \to Pa$, from the Kleisli structure. The Kleisli composition of $f : a \to Pb$ and $g : b \to Pc$, is $g \cdot f = g^\sharp.f : a \to Pc$. This extends to 2-cells so we have composition functors. To obtain a bicategory, it remains to define coherent unit and associativity isomorphisms $\lambda_f : y_b \cdot f \to f$, $\rho_f : f \to f \cdot y_a$ and $\alpha_{h,g,f} : (h \cdot g) \cdot f \to h \cdot (g \cdot f)$. The unit isomorphisms λ_f and ρ_f are given by $(y_b)^\sharp f \xrightarrow{\kappa_b f} 1_{Pb} f \cong f$ and $f \xrightarrow{\eta_f} f^\sharp y_a$ respectively, while the associativity $\alpha_{h,g,f}$ is $(h^\sharp g)^\sharp f \xrightarrow{\kappa_{h,g} f} h^\sharp g^\sharp f \cong h^\sharp(g^\sharp f)$. The coherence axioms follow directly from the coherence conditions of the Kleisli structure.

Theorem 2.2. *Let P be a Kleisli structure on $\mathcal{A} \hookrightarrow \mathcal{K}$. Then $\mathrm{Kl}(P)$ is a bicategory.*

For simplicity in what follows, I shall adopt standard notation and often write $a \nrightarrow b$ instead of $a \to PB$ for maps in $\mathrm{Kl}(P)(a, b)$.

In traditional category theory, the Kleisli construction is one universal way to associate an adjunction with a monad. In the 2-dimensional setting of Kleisli structures we get a restricted (pseudo)adjunction as follows. There is a 'forgetful' pseudofunctor $\mathrm{Kl}(P) \to \mathcal{K}$ taking $f : a \to Pb$ in $\mathrm{Kl}(P)$ to $f^\sharp : Pa \to Pb$ in $\mathcal{K}$. And there is a pseudofunctor $F : \mathcal{A} \to \mathrm{Kl}(P)$, taking $f : a \to b$ in $\mathcal{A}$ to $y_b.f : a \to Pb$ considered as a map from a to b in $\mathrm{Kl}(P)$. I omit the 2-dimensional structure which is routine, but note that the fact that F is a restricted left pseudoadjoint is immediate from the identification $\mathrm{Kl}(P)(a, b) = \mathcal{K}(a, Pb)$.