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Category theoretic structure of setoids

Yoshiki Kinoshita [a,b,*,1], John Power [c,1]

[a] National Institute of Advanced Industrial Science and Technology, 3-11-46, Nakoji, Amagasaki, Hyogo 661-0974, Japan
[b] Department of Information Sciences, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa 259-1293, Japan
[c] Department of Computer Science, University of Bath, Bath BA2 7AY, United Kingdom

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ABSTRACT

A setoid is a set together with a constructive representation of an equivalence relation on it. Here, we give category theoretic support to the notion. We first define a category Setoid and prove it is Cartesian closed with coproducts. We then enrich it in the Cartesian closed category Equiv of sets and classical equivalence relations, extend the above results, and prove that Setoid as an Equiv-enriched category has a relaxed form of equalisers. We then recall the definition of $\mathcal{E}$-category, generalising that of Equiv-enriched category, and show that Setoid as an $\mathcal{E}$-category has a relaxed form of coequalisers. In doing all this, we carefully compare our category theoretic constructs with Agda code for type-theoretic constructs on setoids.

1. Introduction

The notion of setoid, albeit with different nomenclature, was introduced by Bishop in his development of constructive mathematics [1]. The key difference between it and sets is that one does not have equality of elements of a setoid, the closest approximant to equality being given by a constructive representation of an equivalence relation, that is, a family of sets indexed by elements of the setoid. The elements of the family can be regarded as *proof objects* of the relation: the relation is considered to hold if and only if the corresponding set in the family is inhabited. Over recent years, Bishop's idea has been taken up in the field of theorem proving using proof assistants including Agda, Coq and Isabelle [2–4]. Here, we give analysis of the structure of setoids in terms of category theory based on naïve set theory.

The *ordinary* category of setoids and their morphisms is Cartesian closed, but it seems there is no equalisers and co-equalisers; even if they do exist, it would be something strange that cannot be used in a straightforward way. So, we consider *enrichment* over Equiv. The Equiv-category of setoids does have Equiv-inserters, which are weaker notion of equalisers, and cotensors, but it still seems to lack coequalisers and any of its weaker form. We then study the $\mathcal{E}$*-category* of setoids. The $\mathcal{E}$-category **Setoid** does not only have Cartesian closed structure, $\mathcal{E}$-inserters and cotensors, but also $\mathcal{E}$-coinserters and tensors. These are enough to say that there always exists a *weak notion of* limit and colimit of arbitrary (small) diagram in the $\mathcal{E}$-category of setoids. In fact, we give an Agda code which claims the existence in Appendix A.

* Corresponding author at: Department of Information Sciences, Kanagawa University, Tsuchiya 2946, Hiratsuka-shi, Kanagawa, 259-1293, Japan.
E-mail addresses: yoshiki@kanagawa-u.ac.jp (Y. Kinoshita), a.j.power@bath.ac.uk (J. Power).

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We adopt the usual semantic practice of modelling a type by a set and modelling a term in context by a function. The definition of setoid inherently involves a type Set, so we shall assume we have a model of set theory and, with mild overloading of notation, use Set to denote the set of small sets, equivalently a model of sets.

Having adopted those conventions, a setoid A, in classical set-theoretic terms, consists of:

- a set $|A|$
- a family $\approx_A$ of sets indexed by $|A| \times |A|$ (We write $a_0 \approx_A a_1$ for the set indexed by (a_0, a_1).)
- for each $a \in |A|$, an element $\mathsf{refl}_A(a)$ of $a \approx_A a$
- for each pair (a_0, a_1) of elements of $|A|$, a function $\mathsf{sym}_A(a_0, a_1) : (a_0 \approx a_1) \longrightarrow (a_1 \approx a_0)$
- for each triple (a_0, a_1, a_2) of elements of $|A|$, a function $\mathsf{trans}_A(a_0, a_1, a_2) : (a_1 \approx a_2) \times (a_0 \approx a_1) \longrightarrow (a_0 \approx a_2)$

There is some choice about a natural notion of map between setoids, but one natural option, which we shall make, is that a morphism $f : A \longrightarrow B$ consists of:

- a function $\mathsf{fun}_f : |A| \longrightarrow |B|$ together with
- for each pair (a_0, a_1) of elements of $|A|$, a function $\mathsf{resp}_f : (a_0 \approx a_1) \longrightarrow (\mathsf{fun}_f(a_0) \approx \mathsf{fun}_f(a_1))$.

These definitions can be described by the following Agda code.

```
record Setoid : Set₁ where
  field
    carrier : Set
    _≈_ : carrier → carrier → Set
    refl : {x : carrier} → x ≈ x
    sym : {x y : carrier} → x ≈ y → y ≈ x
    trans : {x y z : carrier} → y ≈ z → x ≈ y → x ≈ z

record _⇝_ (A B : Setoid) : Set where
  open Setoid ; _≅_ = _≈_ A ; _≃_ = _≈_ B
  field
    fun : carrier A → carrier B
    resp : {a₀ a₁ : carrier A} → a₀ ≅ a₁ → fun a₀ ≃ fun a₁
```

The most striking fact about the definition of setoids is the absence of coherence axioms. In particular, the data for reflexivity, symmetry and transitivity are exactly data appropriate for the definition of a groupoid: if one added natural coherence axioms to the definition of setoid, one would in fact have the definition of a groupoid. A central idea in the definition of setoid is *not* to insist upon equality between proof objects. The result is that setoids behave quite differently to groupoids or categories.

The behaviour of setoids would be simpler if the sets $a_0 \approx a_1$ were degenerated into singletons or instances of the empty set. That would correspond to the study of the category Equiv of equivalence relations.

The implications of the lack of coherence axioms are profound. For instance, a morphism of setoids, in contrast to a functor, need not preserve the data for reflexivity or transitivity: it follows from the definition of functor that functors preserve n-fold composition for any natural number n, whereas, in the absence of a coherence axiom for transitivity, that would not hold if one imposed the usual functoriality condition on a morphism of setoids. And although we will consider equivalences between morphisms of setoids (cf. natural isomorphisms between functors), it does not make sense to impose a naturality condition on them as, again in the absence of a coherence axiom for transitivity, a composite of such natural transformations would not be natural.

Setoids and morphisms between them generate a category Setoid. The lack of a requirement that the reflexivity, symmetry and transitivity data is preserved by a morphism of setoids impacts on the structure of the category Setoid. If such axioms were imposed on morphisms, the category Setoid would be locally finitely presentable, hence complete and cocomplete. But in fact Setoid seems not to have equalisers, although it does have products and is Cartesian closed.

We will duly study the structure of the category Setoid in this paper, in particular proving that it has products and coproducts and is Cartesian closed: the latter is quite complex. But in theorem proving practice, this category is not of interest per se: constructively, one cannot assert that parallel morphisms $f, g : A \longrightarrow B$ are equal; one can only assert that for each a in $|A|$, the set $f(a) \approx_B g(a)$ is inhabited, i.e., is non-empty. We extend Setoid to provide semantics to express the fact of two morphisms of setoids being equivalent, but not necessarily equal.

In order to provide such structure, we extend Setoid with the canonical structure of an Equiv-enriched category, Equiv being Cartesian closed. We induce an Equiv-enrichment of Setoid from the canonical Equiv-enrichment of Equiv. Cartesian closedness and coproducts extend from Setoid as an ordinary category to **Setoid** as an Equiv-enriched category. We further prove that **Setoid** as an Equiv-enriched category has a relaxed form of equaliser that we call an Equiv-inserter, cf. [5].