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On the number of upward planar orientations of maximal planar graphs [☆]

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ABSTRACT

We consider the problem of determining the maximum and the minimum number of upward planar orientations a maximal planar graph can have. We show that every n-vertex maximal planar graph has at least $\Omega^*(1.189^n)$ and at most $O^*(4^n)$ upward planar orientations. Moreover, we show that there exist n-vertex maximal planar graphs having $O^*(2^n)$ upward planar orientations and n-vertex maximal planar graphs having $\Omega^*(2.599^n)$ upward planar orientations. Further, we present bounds for the maximum and the minimum number of acyclic orientations a maximal planar graph can have.

1. Introduction

A drawing of a graph G in the plane is *upward* if every edge is represented by a y-monotone Jordan curve, and it is *planar* if no two such curves meet other than at common endpoints. An upward drawing induces an acyclic orientation of the edges of G (where each edge is oriented from the vertex with smaller y-coordinate to the one with larger y-coordinate) – this provides a directed acyclic graph $\vec{G}$. An orientation $\vec{G}$ of a planar graph G is *upward planar*, if there exists an upward planar drawing of G that induces $\vec{G}$. In this paper we study the number of possible upward planar orientations of an n-vertex planar graph; in fact, we concentrate on *maximal planar graphs*, also called *triangulations*.

Upward planarity is a natural extension of planarity to directed graphs. When dealing with the visualization of directed graphs, one usually requires an *upward drawing*, i.e., a drawing such that each edge monotonically increases in the y-direction. As a consequence, there has been a lot of work on testing whether a directed graph admits an upward planar drawing (a directed graph that admits such a drawing is called *upward planar graph*) and on constructing upward planar drawings of directed graphs. A non-comprehensive list of remarkable results in the area follows:

- every upward planar graph is a subgraph of a *planar st-graph* [7,15];
- every upward planar graph admits a *straight-line* upward planar drawing [7,15];
- testing the upward planarity of a directed graph is NP-hard [10];
- the upward planarity of a triconnected directed graph can be tested in polynomial time [2]; and
- the upward planarity of a directed graph with a single source can be tested in polynomial time [13].

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Table 1
Bounds for the maximum and the minimum number of upward planar orientations an n-vertex maximal planar graph can have. The $O^*(\cdot)$ and $\Omega^*(\cdot)$ notations hide multiplicative polynomial terms.

	Upper bound	Lower bound
UPO(n)	$O^*(4^n)$	$\Omega^*((23+3\sqrt{57})^{n/4}) = \Omega^*(2.599^n)$
upo(n)	$O^*(2^n)$	$\Omega^*(2^{n/4}) = \Omega^*(1.189^n)$

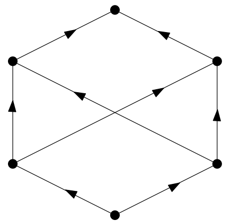

Fig. 1. An acyclic and not upward planar orientation of a planar graph.

Upward drawings have been defined for surfaces other than the plane, such as the *sphere* [12] and the *cylinder* [1].

In this paper we study the minimum and the maximum number of upward planar orientations an n-vertex maximal planar graph can have. We prove the following theorems.

Theorem 1. *Every n-vertex maximal planar graph has at most $O(\sqrt{n}\cdot 4^n)$ upward planar orientations. Moreover, for every $n \geq 3$, there exists an n-vertex maximal planar graph that has $\Omega((23+3\sqrt{57})^{n/4}) = \Omega(2.599^n)$ upward planar orientations.*

Theorem 2. *Every n-vertex maximal planar graph has at least $\Omega(n\cdot 2^{n/4}) = \Omega(n\cdot 1.189^n)$ upward planar orientations. Moreover, for every $n \geq 3$, there exists an n-vertex maximal planar graph that has $O(n\cdot 2^n)$ upward planar orientations.*

Table 1 summarizes the bounds we prove in this paper. Functions UPO(n) and upo(n) represent the maximum and the minimum number of upward planar orientations an n-vertex maximal planar graph can have.

The proof[1] of Theorem 1 relies on a “primal–dual” argument relating the orientations of a maximal planar graph to the orientations of its dual graph and on a counting argument for the latter kind of orientations (Section 3).

The proofs of the lower bound in Theorem 1 and of the upper bound in Theorem 2 are constructive, as they show maximal planar graphs with the claimed number of upward planar orientations (Sections 4 and 5).

The proof of the lower bound in Theorem 2 exploits the decomposition of a maximal planar graph G into outerplanar levels in order to construct, level by level, many upward planar orientations of G (Section 6).

Finally, we study the number of acyclic orientations a maximal planar graph can have (Section 7). Observe that an orientation is upward planar only if it is acyclic. However, there exist acyclic orientations that are not upward planar (see, e.g., the acyclic orientation in Fig. 1). We prove the following theorems.

Theorem 3. *Every n-vertex maximal planar graph has at most $O(5^n)$ acyclic orientations. Moreover, for every $n \geq 3$, there exists an n-vertex maximal planar graph that has $\Omega(4^n)$ acyclic orientations.*

Theorem 4. *Every n-vertex maximal planar graph has at least $\Omega(2^n)$ acyclic orientations. Moreover, for every $n \geq 3$, there exists an n-vertex maximal planar graph that has $O(4^n)$ acyclic orientations.*

While counting the number of upward planar orientations of a planar graph is, as far as we know, a novel research challenge, counting the number of acyclic orientations of a graph is a very well-studied topic of research. Stanley [22] proved that the number of acyclic orientations of an n-vertex graph G is equal to $(-1)^n P(G,-1)$, where $P(G,k)$ is the *chromatic polynomial* of a graph G, that is the polynomial with degree n representing the number of ways of k-coloring the vertices of G; the chromatic polynomial is a special case of the famous *Tutte polynomial* [3]. Linial [16] showed that determining the number of acyclic orientations of a given graph is $\#P$-complete. An upper bound on the number of acyclic orientations of any graph G has been proved by Fredman [11], and Manber and Tompa [17]; the upper bound reads as $\prod_{v\in G}(d(v)+1)$, where $d(v)$ is the degree of a vertex v in G. Kahale and Schulman [14] proved an upper bound of $\prod_{i=1}^{n}(\gamma_i+1)$ for the number of acyclic orientations of any graph G, where $\{\gamma_i\}$ are the eigenvalues of the Laplacian of G. To the best of our knowledge, no non-trivial upper and lower bounds for the maximum and minimum number (as a function of n) of acyclic orientations of a planar graph were known previous to this work.

[1] In the conference version of the paper, we exhibited a different proof of Theorem 1 relying on a “canonical ordering” for maximal upward planar graphs introduced by Mehlhorn in [18] and on a counting argument that exploits such a canonical ordering.