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Characterization, definability and separation via saturated models

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ABSTRACT

Three important results about the expressivity of a modal logic $\mathfrak{L}$ are the Characterization Theorem (that identifies a modal logic $\mathfrak{L}$ as a fragment of a better known logic), the Definability Theorem (that provides conditions under which a class of $\mathfrak{L}$-models can be defined by a formula or a set of formulas of $\mathfrak{L}$), and the Separation Theorem (that provides conditions under which two disjoint classes of $\mathfrak{L}$-models can be separated by a class definable in $\mathfrak{L}$).

We provide general conditions under which these results can be established for a given choice of model class and modal language whose expressivity is below first order logic. Besides some basic constraints that most modal logics easily satisfy, the fundamental condition that we require is that the class of ω-saturated models in question has the Hennessy–Milner property with respect to the notion of observational equivalence under consideration. Given that the Characterization, Definability and Separation theorems are among the cornerstones in the model theory of $\mathfrak{L}$, this property can be seen as a test that identifies the adequate notion of observational equivalence for a particular modal logic.

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1. Introduction

Syntactically, modal languages [7] are propositional languages extended with *modal operators*. Indeed, the basic modal language is defined as the extension of the propositional language with the unary operator $\Diamond$. Although these languages have a very simple syntax, they are extremely useful to describe and reason about *relational structures*. A relational structure is a nonempty set together with a family of n-ary relations. Given the generality of this definition it is not surprising that modal logics are used in a wide range of disciplines: mathematics, philosophy, computer science, computational linguistics, etc. For example, in theoretical computer science, labeled transition systems (which are nothing but relational structures) are used to model the execution of a program.

An important observation that might have gone unnoticed in the above paragraphs is that we talk about modal *logics*, in plural. There is, nowadays, a wide variety of modal languages and an extensive menu of modal operators to choose from: *Since* and *Until* [19], universal modality [15], difference modality [11], fix-point operators [21], are some of the possibilities to name only a few. This multiplicity is both a boon and a bane. On the one hand, the variety comes in handy when we need to choose the proper logic to model a particular problem. But it also means that many results have to be established time

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and again for each new logic that arrives in town. It is here when a solid model theory is useful. With the proper theoretical tools, some results might be established just by verifying certain properties of the class of models defining the logic. In particular, many model theoretical results for a logic $\mathfrak{L}$ rely on the availability of an adequate notion of "indiscernibility" or observational equivalence, i.e., a notion that specifies when two models are indistinguishable by formulas of $\mathfrak{L}$.

We investigate Characterization, Definability, and Separation theorems for modal logics: three model-theoretical results intimately related with the notion of observational equivalence. We pursue a general study of these properties without referring to a particular modal logic. In general, the validity of these theorems is a good indicator that the underlying notion of observational equivalence for a given logic is indeed the correct one.

First-order logic, modal logics and similarity These three notions will play a mayor role and it will be useful to discuss them and their interaction right away. First-order logic (FO) will delineate our framework and we will assume its syntax, semantics and basic properties well known.[1] All the modal logics covered by our results are fragments of FO, and we will make use of some of FO's main model theoretic properties to prove our results. We will introduce the basic (uni)modal logic BML in detail but, in the rest of this paper we will work with an arbitrary modal logic. We will only require it to be *adequately below first-order logic* as per Definition 1. Finally, we will discuss different notions of observational equivalence. They will depend on the particular logic under consideration but, once more, we will abstract away their common aspects in the notion of an *adequate similarity* as per Definition 3.

Let us start by introducing syntax and semantics of BML. Let PROP be a countable, infinite set of propositional symbols. Formulas in BML are generated by the grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi,$$

where p is a propositional symbol in PROP. BML is interpreted over relational models $\mathcal{M} = (M, R, V)$, where M is a nonempty domain, R is a binary relation on M, and V is a valuation mapping propositional symbols to subsets over M. The pair $\langle \mathcal{M}, w\rangle$ for w an element in M is called a pointed model. We usually drop brackets and write $\mathcal{M}, w$ instead of $\langle \mathcal{M}, w\rangle$. Given a pointed model $\mathcal{M}, w$ we define when a BML-formula φ is true in $\mathcal{M}$ at w (notation $\mathcal{M}, w \Vdash \varphi$) as follows:

$$\begin{array}{rll} \mathcal{M}, w \Vdash p & \text{iff} & w \in V(p) \\ \mathcal{M}, w \Vdash \neg\varphi & \text{iff} & \mathcal{M}, w \nVdash \varphi \\ \mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \Diamond\varphi & \text{iff} & \mathcal{M}, v \Vdash \varphi \text{ for some } v \in M \text{ such that } wRv. \end{array}$$

The right notion of observational equivalence for BML is that of a *bisimulation*. A bisimulation between two models $\mathcal{M} = (M, R, V)$ and $\mathcal{M}' = (M', R', V')$ is a nonempty relation $Z \subseteq M \times M'$ satisfying the following conditions:

(i) *Atomic harmony*: if wZw' then w and w' satisfy the same propositional symbols, i.e., $w \in V(p)$ iff $w' \in V'(p)$ for all propositional symbols p;
(ii) *Forth condition*: if wZw' and wRv then there is v' s.t. vZv' and $w'R'v'$;
(iii) *Back condition*: if wZw' and $w'R'v'$ then there is v s.t. vZv' and wRv.

Two pointed models $\mathcal{M}, w$ and $\mathcal{M}', w'$ are called *bisimilar* if there is a bisimulation Z between M and M' such that wZw'. A well known result in basic modal logic states that if $\mathcal{M}, w$ and $\mathcal{M}', w'$ are bisimilar then they are *modally equivalent*, i.e., for any BML-formula φ we have $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}', w' \Vdash \varphi$. The reverse implication is not true in general. A model $\mathcal{M}$ is called modally-saturated if for every state $w \in M$ and every set Σ of formulas, if every finite subset of Σ is satisfiable in some successor of w, then Σ itself is satisfiable in some successors of w. An important result states that if two modally saturated models are modally equivalent then they are bisimilar [7].

We now switch to first-order logic. Notice, first, that a relational model $\mathcal{M} = (M, R, V)$ is essentially a first-order model over the language with a binary relation symbol and unary predicate symbols for the propositional symbols. Second, bisimulations are the modal analogue of the first-order notion of *partial isomorphism*. That is, partial isomorphisms are the right notion of observational equivalence for FO. Given a model $\mathcal{M}$ and $w_1, \ldots, w_n$ elements in M, we write $(\mathcal{M}, w_1, \ldots, w_n)$ for the extension of $\mathcal{M}$ with $w_1, \ldots, w_n$ as new constant symbols (interpreted in the obvious way). A partial isomorphism between two first-order models $\mathcal{M}$ and $\mathcal{M}'$ is a binary relation Z on pairs of finite sequences $\langle w_1, \ldots, w_n\rangle$, $\langle w'_1, \ldots, w'_n\rangle$ of elements of M and M' of the same length such that $\emptyset Z \emptyset$ and

(i) *Atomic harmony*: if $\langle w_1, \ldots, w_n\rangle Z \langle w'_1, \ldots, w'_n\rangle$ then $(\mathcal{M}, w_1, \ldots, w_n)$ and $(\mathcal{M}', w'_1, \ldots, w'_n)$ satisfy the same atomic sentences;

[1] We will use standard notation for first-order models and formulas and, in particular, we will use $\models$ for the satisfiability relation between a first-order model $\mathcal{M}$, an assignment g and a first-order formula φ.

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