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On lattices from combinatorial game theory modularity and a representation theorem: Finite case ☆

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ABSTRACT

We show that a self-generated set of combinatorial games, S, may not be hereditarily closed but, strong self-generation and hereditary closure are equivalent in the universe of short games. In [13], the question "Is there a set which will give a non-distributive but modular lattice?" appears. A useful necessary condition for the existence of a finite non-distributive modular $\mathcal{L}(S)$ is proved. We show the existence of S such that $\mathcal{L}(S)$ is modular and not distributive, exhibiting the first known example. More, we prove a *Representation Theorem with Games* that allows the generation of all finite lattices in game context. Finally, a computational tool for drawing lattices of games is presented.

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1. Introduction

Combinatorial game theory (CGT) studies perfect information games in which there are no chance devices (e.g. dice) and two players take turns moving alternately. This paper concerns itself with games under normal play, where last player to move wins. This paper will be self-contained but see [1,5,8] for background (see [9] for a survey).

The players are traditionally called Left (or L) and Right (or R). The options of a game are all those positions which can be reached in one move. In CGT, games can be expressed recursively as $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L$ are the Left options and $\mathcal{G}^R$ are the Right options of G. The followers of G are all the games that can be reached by some sequence of moves from G. Conway gave a recursive construction based on a transfinite sequence of days [8]. For example, starting with the empty set, the game $\{\,\mid\,\} = 0$ is born on day zero; the games 0, $\{0\mid\,\} = 1$, $\{0 \mid 0\} = *$, $\{\,\mid 0\} = -1$ are born on day one; and there are 22 games born on day two; 1474 games are born on day three; and somewhere between 3 trillion and 10^{434} games are born on day four (see [10]). Games with a finite number of followers, and hence the depth of the game tree is finite, are called *short games* and all are born before the day ω. The games born on day ω or after are called *long* games. The exact number of games born by day $n < \omega$ or even a good asymptotic formula is not known. Fortunately, the number of games is not the

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issue for us. Games also form a partial order and we investigate the lattices that can occur. In 2002, Calistrate, Paulhus and Wolfe [6] proved:

Theorem 1. *(See [6].) The set of games born on day n is a distributive lattice.*

These are the games generated recursively from the empty set. What happens if the original set is something else?

Definition 1. Let S be a set of games. The set of children of S, $\mathcal{L}(S)$, is the set of games $\{G\colon\ G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L, \mathcal{G}^R \subseteq S\}$. Let $\lfloor G \rfloor = \{J \in S\colon\ J \lhd\!\!\mid G\}$ and $\lceil G \rceil = \{J \in S\colon\ J \mid\!\!\rhd G\}$.

For completeness, we give the facts and definitions about games needed for this paper.

Games with a finite number of followers, and hence the depth of the game tree is finite, are called *short* games and all are born before the day ω. The games born on day ω or after are called *long* games. Conway's inductive definition constructs the complete set of combinatorial games and, in the process, gives an alternative construction of real numbers not based on the rational numbers.

In the disjunctive sum, $G + H$ of games G and H a player is allowed to move in just one of the summands, i.e., $G + H = \{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\}$. The set of combinatorial games (short and long) with disjunctive sum is an abelian group. By convention, Left is associated with positive and Right is associated with negative. Every game, G, has an additive inverse $-G = \{-\mathcal{G}^R \mid -\mathcal{G}^L\}$.

In the following table, "Previous" and "Next" refer to the *previous player* and *next player* respectively.

	Meaning		Meaning
$G > 0$	Left wins G	$G > H$	Left wins $G - H$
$G = 0$	Previous wins G	$G = H$	Previous wins $G - H$
$G < 0$	Right wins G	$G < H$	Right wins $G - H$
$G \parallel 0$	Next wins G	$G \parallel H$	Next wins $G - H$
$G \geqslant 0$	$G = 0$ or $G > 0$ (Left wins going second)	$G \geqslant H$	$G = H$ or $G > H$ (Left wins going second in $G - H$)
$G \mid\!\!\rhd 0$	$G \parallel 0$ or $G > 0$ (Left wins going first)	$G \mid\!\!\rhd H$	$G \parallel H$ or $G > H$ (Left wins going first in $G - H$)
$G \leqslant 0$	$G = 0$ or $G < 0$ (Right wins going second)	$G \leqslant H$	$G = H$ or $G < H$ (Right wins going second in $G - H$)
$G \lhd\!\!\mid 0$	$G \parallel 0$ or $G < 0$ (Right wins going first)	$G \lhd\!\!\mid H$	$G \parallel H$ or $G < H$ (Right wins going first in $G - H$)

The set of games forms an equivalence relation under $=$ and a partial order under $>$. Note that if $G \parallel H$ then, in the partial order, G and H are incomparable. In the universe of short games, there is a unique representative in each equivalence class, called the *canonical form*.

The canonical form of a short game is found by (i) eliminating dominated options, that is if $A > B$ then Left prefers A and would eliminate B from consideration; the converse for Right; and (ii) by-passing reversible options, that is if there is a Left option A of a game G with a Right option B of A such that $B \leqslant G$ then A can be replaced by all the left options of B, again the converse for Right. Intuitively: (i) don't play bad moves; (ii) if you give your opponent a move that allows him a move to a better game than the original, assume that they will make it.

Theorem 2. *(See [1,5,8].) Let $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ be a short game. Then G is in canonical form iff* (i) *$H, K \in \mathcal{G}^L$, $H \neq K$, then $H \parallel K$ (similarly for $H, K \in \mathcal{G}^R$), and* (ii) *for each $K \in \mathcal{H}^R$, where $H \in \mathcal{G}^L$, we have $K \mid\!\!\rhd G$ (similarly for $K \in \mathcal{H}^L$ where $H \in \mathcal{G}^R$).*

In 2012, the ordered structure of the games of $\mathcal{L}(S)$ for an arbitrary set S of games was studied and the following results were proved (we combine them in one).

Theorem 3. *(See Albert and Nowakowski [2].) For any set of games S we have*

- *$\mathcal{L}(S)$ is a complete lattice;*
- *if $G \in \mathcal{L}(S)$ then $G = \{\lfloor G \rfloor \mid \lceil G \rceil\}$;*
- *$G \vee H = \{\lfloor G \rfloor \cup \lfloor H \rfloor \mid \lceil G \rceil \cap \lceil H \rceil\}$;*
- *$G \wedge H = \{\lfloor G \rfloor \cap \lfloor H \rfloor \mid \lceil G \rceil \cup \lceil H \rceil\}$.*

In the same paper [2], the authors presented the following definition and theorem.

Definition 2. Let S be a set of short games. S is hereditarily closed if for each $G \in S$ all the options in the canonical form of G belong to S.

Observation 1. We observe that hereditary closure is a property of a set of *short* games because it needs the concept of canonical form. Also, knowing the canonical forms of the games, it is not needed to think about $\mathcal{L}(S)$ to verify if S has the property.

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