Theoretical Computer Science 536 (2014) 42–61

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# Weak morphisms of higher dimensional automata ☆

Thomas Kahl

*Centro de Matemática, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal*

ARTICLE INFO

*Article history:*
Received 8 March 2013
Received in revised form 26 November 2013
Accepted 21 February 2014
Communicated by R. van Glabbeek

*Keywords:*
Higher dimensional automata
Precubical set
Geometric realisation
Trace language
Preorder relation
Abstraction

ABSTRACT

We introduce weak morphisms of higher dimensional automata and use them to define preorder relations for HDAs, among which are homeomorphic abstraction and trace equivalent abstraction. It is shown that homeomorphic abstraction is essentially always stronger than trace equivalent abstraction. We also define the trace language of an HDA and show that, for a large class of HDAs, it is invariant under trace equivalent abstraction.


## Introduction

One of the most expressive models of concurrency is the one of higher dimensional automata [14]. A higher dimensional automaton (HDA) over a monoid $M$ is a precubical set with initial and final states and with 1-cubes labelled by elements of $M$ such that opposite edges of 2-cubes have the same label. Intuitively, an HDA can be seen as an automaton with an independence relation represented by cubes. If two actions $a$ and $b$ are enabled in a state $q$ and are independent in the sense that they may be executed in any order or even simultaneously without any observable difference, then the HDA contains a 2-cube linking the two execution sequences $ab$ and $ba$ beginning in $q$. Similarly, the independence of $n$ actions is represented by $n$-cubes. It has been shown in [14] that many classical models of concurrency can be translated into the one of HDAs. HDA semantics for process algebras are given in [10] and [17].

In this paper, we introduce three preorder relations for HDAs. Whenever, as in Fig. 1, an HDA $\mathcal{B}$ is a subdivision of an HDA $\mathcal{A}$, then $\mathcal{A}$ is related to $\mathcal{B}$ in each of these preorders.

The definitions of the preorder relations are based on the concept of weak morphism, which is developed in Section 2. Roughly speaking, a weak morphism between two HDAs is a continuous map between their geometric realisations that sends subdivided cubes to subdivided cubes and that preserves labels of paths. A morphism of HDAs, or, more precisely, its geometric realisation, is a weak morphism but not vice versa. For example, in Fig. 1, there exists a weak morphism from $\mathcal{A}$ to $\mathcal{B}$, but there does not exist any morphism between the two HDAs. If there exists a weak morphism from an HDA $\mathcal{A}$ to an HDA $\mathcal{B}$, we write $\mathcal{A} \to \mathcal{B}$. The relation $\to$ is the first preorder relation for HDAs we consider in this paper. It has the basic property that $\mathcal{A} \to \mathcal{B}$ implies that the language (the behaviour) of $\mathcal{A}$ is contained in the language of $\mathcal{B}$. One may consider

☆ This research has been supported by FEDER funds through "Programa Operacional Factores de Competitividade – COMPETE" and by FCT – *Fundação para a Ciência e a Tecnologia* through projects Est-C/MAT/UI0013/2011 and PTDC/MAT/0938317/2008.

*E-mail address:* kahl@math.uminho.pt.

http://dx.doi.org/10.1016/j.tcs.2014.02.028
0304-3975/


(a) $\mathcal{A}$ (b) $\mathcal{B}$

**Fig. 1.** Two HDAs $\mathcal{A}$ and $\mathcal{B}$ over the free monoid on $\{a, b, c\}$ such that $\mathcal{B}$ is a subdivision of $\mathcal{A}$, $\mathcal{A} \to \mathcal{B}$, $\mathcal{A} \stackrel{\sim_t}{\to} \mathcal{B}$ and $\mathcal{A} \stackrel{\approx}{\to} \mathcal{B}$.

$\to$ as a kind of weak simulation preorder: If there exists a weak morphism from $\mathcal{A}$ to $\mathcal{B}$, then for every 1-cube in $\mathcal{A}$ with label $\alpha$ from a vertex $v$ to a vertex $w$ there exists a path in $\mathcal{B}$ with label $\alpha$ from the image of $v$ to the image of $w$.

The higher dimensional structure of an HDA induces an independence relation on the monoid of labels and allows us to define the trace language of an HDA in Section 3. The fundamental concept in this context is dihomotopy (short for directed homotopy) of paths [8,15]. Besides the trace language of an HDA, we also consider the trace category of an HDA and trace equivalences between HDAs. The trace category of an HDA is a variant of the fundamental bipartite graph of a d-space [3]. Its objects are certain important states of the HDA, including the initial and the final ones, and the morphisms are the dihomotopy classes of paths between these states. A trace equivalence is essentially defined as a weak morphism that induces an isomorphism of trace categories. If there exists a trace equivalence from an HDA $\mathcal{A}$ to an HDA $\mathcal{B}$, we say that $\mathcal{A}$ is a trace equivalent abstraction of $\mathcal{B}$ and write $\mathcal{A} \stackrel{\sim_t}{\to} \mathcal{B}$. Trace equivalent abstraction is our second preorder relation for HDAs. We show that for a large class of HDAs, $\mathcal{A} \stackrel{\sim_t}{\to} \mathcal{B}$ implies that there exists a bijection between the trace languages of $\mathcal{A}$ and $\mathcal{B}$.

The third preorder relation is called homeomorphic abstraction and is the subject of Section 4. We say that an HDA $\mathcal{A}$ is a homeomorphic abstraction of an HDA $\mathcal{B}$ and write $\mathcal{A} \stackrel{\approx}{\to} \mathcal{B}$ if there exists a weak morphism from $\mathcal{A}$ to $\mathcal{B}$ that is a homeomorphism and a bijection on initial and on final states. Homeomorphic abstraction may be seen as a labelled version of T-homotopy equivalence in the sense of [13,11]. We show that under a mild condition, $\mathcal{A} \stackrel{\approx}{\to} \mathcal{B}$ implies $\mathcal{A} \stackrel{\sim_t}{\to} \mathcal{B}$.

## 1. Precubical sets and HDAs

This section contains some basic and well-known material on precubical sets and higher dimensional automata.

### 1.1. Precubical sets

A *precubical set* is a cubical set [1,20,21] without degeneracies, i.e. a graded set $P = (P_n)_{n \geqslant 0}$ with *boundary operators* $d_i^k : P_n \to P_{n-1}$ $(n > 0,\ k = 0, 1,\ i = 1, \dots, n)$ satisfying the relations $d_i^k \circ d_j^l = d_{j-1}^l \circ d_i^k$ $(k, l = 0, 1,\ i < j)$ [5,7,8,13,15] (cf. also [25]). The least $n \geqslant 0$ such that $P_i = \emptyset$ for all $i > n$ is called the *dimension* of $P$. If no such $n$ exists, then the dimension of $P$ is $\infty$. If $x \in P_n$, we say that $x$ is of *degree* $n$ and write $\deg(x) = n$. The elements of degree $n$ are called the *n-cubes* of $P$. The elements of degree 0 are also called the *vertices* or the *nodes* of $P$. A morphism of precubical sets is a morphism of graded sets that is compatible with the boundary operators.

The category of precubical sets can be seen as the presheaf category of functors $\square^{\mathrm{op}} \to \mathbf{Set}$ where $\square$ is the small subcategory of the category of topological spaces whose objects are the standard $n$-cubes $[0, 1]^n$ $(n \geqslant 0)$ and whose non-identity morphisms are composites of the maps $\delta_i^k : [0, 1]^n \to [0, 1]^{n+1}$ $(k \in \{0, 1\},\ n \geqslant 0,\ i \in \{1, \dots, n + 1\})$ given by $\delta_i^k(u_1, \dots, u_n) = (u_1, \dots, u_{i-1}, k, u_i, \dots, u_n)$. Here, we use the convention that given a topological space $X$, $X^0$ denotes the one-point space $\{(\ )\}$.

### 1.2. Precubical subsets

A *precubical subset* of a precubical set $P$ is a graded subset of $P$ that is stable under the boundary operators. It is clear that a precubical subset is itself a precubical set. Note that unions and intersections of precubical subsets are precubical subsets and that the image of a morphism $f : P \to Q$ of precubical sets is a precubical subset of $Q$.

### 1.3. Intervals

Let $k$ and $l$ be integers such that $k \leqslant l$. The *precubical interval* $[\![k, l]\!]$ is the at most 1-dimensional precubical set defined by $[\![k, l]\!]_0 = \{k, \dots, l\}$, $[\![k, l]\!]_1 = \{[k, k + 1], \dots, [l - 1, l]\}$, $d_1^0[j - 1, j] = j - 1$ and $d_1^1[j - 1, j] = j$.