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Fault tolerance in the arrangement graphs ☆

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ABSTRACT

Let n and k be positive integers with $n-k \geqslant 1$. The arrangement graph $A_{n,k}$ is recognized as an attractive interconnection network. Let f_m be the minimum number of faulty vertices that make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model. In this paper, we prove that $f_0 = 1$, $f_1 = n$, $f_{n-2} = n!/2$, and $n!/(n-m)! \leqslant f_m \leqslant \binom{k-1}{m-1}n!/(n-m)! - 2\binom{k-2}{m-1}n!/(n-m+1)!$ for $2 \leqslant m \leqslant k-1$.

1. Introduction

The study of interconnection networks has been an important research area for parallel and distributed computer systems. It is well known that an interconnection network is usually represented by an undirected simple graph G. We denote the *vertex set* and the *edge set* of G by $V(G)$ and $E(G)$, respectively. As failures are inevitable, fault-tolerance of interconnection networks has become an important issue and has been extensively studied (see, for example, [4,6,8,10,12–14]).

The fault tolerance of interconnection networks is generally measured by how much of the network structure is preserved in the presence of a given number of vertex and/or edge failures. Parallel algorithms running on the networks utilize the topological properties of these networks. Obviously, in the presence of component failures, the entire interconnection network is not available. Thus the natural question is how large of a *subnetwork* (defined as a smaller network but with the same topological properties as the original one) is still available in the faulty network. Under this consideration, Becker and Simon [4] studied the minimum number of faults, necessary for an adversary to destroy each $(n-k)$-dimensional subcube in an n-dimensional hypercube. Latifi [10] presented a bound on the number of faulty vertices to make every $(n-k)$-dimensional substar faulty in an n-dimensional star graph and also determined the exact value for some special cases. Wang and Yang [12] investigated the minimum number of faulty vertices to make every $(n-k)$-dimensional sub-bubble-sort graph faulty in an n-dimensional bubble-sort graph. Subsequently, this problem was also studied by Wang et al. [13] for k-ary n-cube networks.

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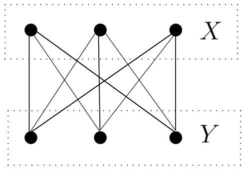

Fig. 1. A balanced bipartite graph.

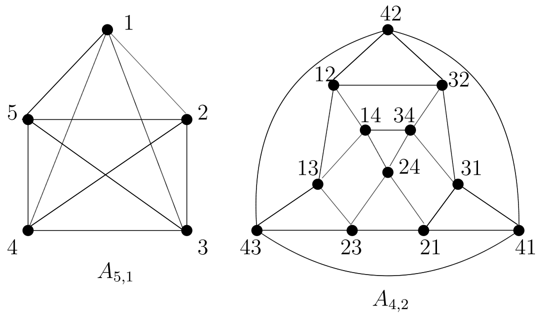

Fig. 2. $A_{5,1}$ and $A_{4,2}$.

The interconnection network considered in this paper is the *arrangement graph*, denoted by $A_{n,k}$, which was proposed by Day and Tripathi [7] as a generalization of the star graph. It is more flexible in its size than the star graph. Since the arrangement graph has been proved to possess many attractive properties such as regularity, vertex symmetry and edge symmetry, it has drawn considerable research attentions recently [5,6,9,11,14]. In this paper, we are interested in the minimum number f_m of faulty vertices to make every sub-arrangement graph $A_{n-m,k-m}$ faulty in $A_{n,k}$ under vertex-failure model. We prove that $f_0 = 1$, $f_1 = n$, $f_{n-2} = n!/2$, and $n!/(n-m)! \leqslant f_m \leqslant \binom{k-1}{m-1} n!/(n-m)! - 2\binom{k-2}{m-1} n!/(n-m+1)!$ for $2 \leqslant m \leqslant k-1$. The rest of this paper is organized as follows. In Section 2, we introduce the arrangement graph and some of its properties. In Section 3, we prove the main results. Conclusions are covered in Section 4.

2. Preliminaries

In the remainder of this paper, we follow [3] for the graph-theoretical terminology and notation not defined here. A graph is called a *balanced bipartite graph* if its vertex set can be partitioned into two subsets X and Y with $|X| = |Y|$ so that every edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph. Fig. 1 shows the diagram of a balanced bipartite graph. Let G and H be two graphs. G and H are *distinct* if their vertex sets are different. G and H are *disjoint* if their vertex sets have no common vertex. Two edges in $E(G)$ are *independent* if they are nonadjacent in G. Given a positive integer n, let $\langle n\rangle$ denote the set $\{1, 2, \dots, n\}$.

Assume that n and k are two positive integers with $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. The *vertex set* of the arrangement graph $A_{n,k}$, $V(A_{n,k}) = \{u\colon\ u = u_1u_2\dots u_k$ with $u_i \in \langle n\rangle$ for $1 \leqslant i \leqslant k$ and $u_i \neq u_j$ if $i \neq j\}$, and two vertices $u = u_1u_2\dots u_k$ and $v = v_1v_2\dots v_k$ are *adjacent* if they differ in exactly one position j, where $j \in \langle k\rangle$. Such an edge (u, v) is called a *j-edge*. By definition, $A_{n,k}$ is a regular graph of degree $k(n-k)$ with $n!/(n-k)!$ vertices. Moreover, $A_{n,1}$ is isomorphic to the complete graph K_n. Indeed, $A_{n,n-1}$ is isomorphic to the n-dimensional star graph S_n [1], and $A_{n,n-2}$ is isomorphic to the n-alternating group graph AG_n [5]. $A_{5,1}$ and $A_{4,2}$ are shown in Fig. 2.

A standard way to view $A_{n,k}$ is via its recursive structure. Let i and j be two positive integers with $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$. Let $H_{i,j}$ be the subgraph of $A_{n,k}$ induced by the vertex set $\{u\colon\ u = u_1u_2\dots u_k \in V(A_{n,k})$ and $u_i = j\}$. Then $H_{i,j}$ is isomorphic to $A_{n-1,k-1}$. Thus, $A_{n,k}$ can be recursively constructed from n copies of $A_{n-1,k-1}$, and we say that it is a *decomposition* via the i-th position. Every vertex $v = v_1v_2\dots v_k$ in $H_{i,j}$ has exactly $n-k$ neighbors outside of $H_{i,j}$; moreover, its $n-k$ neighbors belong to distinct $H_{i,l}$'s, where $l \in \langle n\rangle\backslash\{v_1, v_2, \dots, v_k\}$. We call these neighbors the *external neighbors* of v. It is easy to see that the edges whose end-vertices belong to distinct $H_{i,j}$'s are i-edges. For a given pair of $H_{i,j}$ and $H_{i,l}$ with $j \neq l$, there are $(n-2)!/(n-k-1)!$ i-edges between them; moreover, these i-edges are independent. For example, $A_{4,2}$ can be decomposed into $H_{2,1}, H_{2,2}, H_{2,3}, H_{2,4}$ via the 2-nd position. For any $j \in \langle 4\rangle$, $H_{2,j}$ is isomorphic to $A_{3,1}$; moreover, $H_{2,1}$, $H_{2,2}$, $H_{2,3}$ and $H_{2,4}$ are the triangles $(21, 31, 41, 21)$, $(12, 32, 42, 12)$, $(13, 23, 43, 13)$ and $(14, 24, 34, 14)$, respectively (see Fig. 2).

Given two integers $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$, for any integer m $(0 \leqslant m \leqslant k-1)$, let $i_1, i_2, \dots, i_m$ be m integers with $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant k$ and let $a_{i_1}, a_{i_2}, \dots, a_{i_m}$ be m pairwise distinct integers in $\langle n\rangle$. Denote $M = \{b_1b_2\dots b_{i_1-1}a_{i_1}b_{i_1+1}\dots b_{i_2-1}a_{i_2}b_{i_2+1}\dots a_{i_m}b_{i_m+1}\dots b_k\colon\ b_1, b_2, \dots, b_{i_1-1}, b_{i_1+1}, \dots, b_{i_2-1}, b_{i_2+1}, \dots, b_{i_m+1}, \dots, b_k \in \langle n\rangle\backslash\{a_{i_1}, a_{i_2}, \dots, a_{i_m}\}$ and they are pairwise distinct$\}$. In particular, $b_1b_2\dots b_{i_1-1}$ and $b_{i_m+1}\dots b_k$ are empty strings if $i_1 = 1$ and $i_m = k$, respectively. Obviously, the subgraph of $A_{n,k}$ induced by M is isomorphic to $A_{n-m,k-m}$. Let X be a *don't care*