Theoretical Computer Science 529 (2014) 96–110

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Characterizations of non-associative ordered semigroups by their fuzzy bi-ideals

Tariq Shah, Nasreen Kausar

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

ARTICLE INFO

Article history:
Received 18 March 2012
Received in revised form 9 October 2013
Accepted 2 February 2014
Communicated by A. Skowron

Keywords:
Fuzzy (set, AG-subgroupoids)
Fuzzy left (resp. right, bi-, generalized bi-, (1, 2)-) ideals

ABSTRACT

The aim of this paper is to investigate the characterizations of different classes of non-associative and non-commutative ordered semigroups in terms of fuzzy left (right, bi-, generalized bi-, $(1,2)$-) ideals.

1. Introduction

In 1972, a generalization of commutative semigroups has been established by Kazim and Naseeruddin [3]. In ternary commutative law: $abc = cba$, they introduced the braces on the left side of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is called the left invertive law. A groupoid S is said to be a left almost semigroup (abbreviated as LA-semigroup) if it satisfies the left invertive law: $(ab)c = (cb)a$. This structure is also known as Abel–Grassmann's groupoid (abbreviated as AG-groupoid) in [13]. An AG-groupoid is a midway structure between an abelian semigroup and a groupoid. Mushtaq and Yusuf [12] investigated the concept of ideals in AG-groupoids.

In [2] (resp. [1]), a groupoid S is said to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [3], an AG-groupoid is medial, but in general an AG-groupoid needs not to be paramedial. Every AG-groupoid with left identity is paramedial by Protic and Stevanovic [13] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

In [4], if $(S, \cdot, \leqslant)$ is an ordered semigroup and $\emptyset \neq A \subseteq S$, we define a subset of S as follows: $(A] = \{s \in S\colon\ s \leqslant a \text{ for some } a \in A\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq A$.

The notions of ideals play a crucial role in the study of the (ring, semiring, near-ring, semigroup, ordered semigroup) theory, etc.

A non-empty subset A of S is called a left (resp. right) ideal of S if the following hold: (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$).

A subsemigroup (a non-empty subset) A of S is called a bi- (generalized bi-) ideal of S if (1) $ASA \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$. Every bi-ideal of S is a generalized bi-ideal of S. A subsemigroup A of S is called a $(1,2)$-ideal of S if (1) $ASA^2 \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$.

E-mail addresses: stariqshah@gmail.com (T. Shah), kausar.nasreen57@gmail.com (N. Kausar).

http://dx.doi.org/10.1016/j.tcs.2014.02.007
0304-3975/

In [4,6], an ordered semigroup S is said to be a regular if for every $a \in S$ there exists an element $x \in S$ such that $a \leqslant axa$. Equivalent definitions are as follows: (1) $A \subseteq (ASA]$ for every $A \subseteq S$. (2) $a \in (aSa]$ for every $a \in S$.

An ordered semigroup S is said to be a $(2,2)$-regular if for every $a \in S$, there exists an element $x \in S$ such that $a \leqslant a^2xa^2$. Equivalent definitions are as follows: (1) $A \subseteq (A^2SA^2]$ for every $A \subseteq S$. (2) $a \in (a^2Sa^2]$ for every $a \in S$.

In [5,6], an ordered semigroup S is said to be an intra-regular if for every $a \in S$ there exist elements $x, y \in S$ such that $a \leqslant xa^2y$. Equivalent definitions are as follows: (1) $A \subseteq (SA^2S]$ for every $A \subseteq S$. (2) $a \in (Sa^2S]$ for every $a \in S$.

The idea of ordering of AG-groupoids has been initiated in [15]. Shah et al. [16] have investigated the concept of m-(resp. n-, i-) systems in ordered AG-groupoids.

We will initiate the concept of regular (resp. left regular, right regular, $(2,2)$-regular, left weakly regular, right weakly regular, intra-regular) ordered AG-groupoids. Despite the fact that structure (ordered AG-groupoid) is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

We will describe a study of regular (resp. left regular, right regular, $(2,2)$-regular, left weakly regular, right weakly regular, intra-regular) ordered AG-groupoids by the properties of fuzzy left (right, bi-, generalized bi-, $(1,2)$-) ideals. In this regard, we will prove that in regular (resp. left weakly regular) ordered AG-groupoids, the concept of fuzzy (right, two-sided) ideals coincides. We will also show that in right regular (resp. $(2,2)$-regular, right weakly regular, intra-regular) ordered AG-groupoids, the concept of fuzzy (left, right, two-sided) ideals coincides. Also in left regular ordered AG-groupoids with left identity, the concept of fuzzy (left, right, two-sided) ideals coincides. We will also characterize left weakly regular ordered AG-groupoids in terms of fuzzy right (two-sided, bi-, generalize bi-) ideals.

2. Some basic definitions and preliminary results

An ordered AG-groupoid S is a partially ordered set, at the same time an AG-groupoid, such that $a \leqslant b$, implies $ac \leqslant bc$ and $ca \leqslant cb$ for all $a, b, c \in S$. Two conditions are equivalent to the one condition $(ca)d \leqslant (cb)d$ for all $a, b, c, d \in S$.

Let S be an ordered AG-groupoid and $\emptyset \neq A \subseteq S$, we define a subset $(A] = \{s \in S\colon\ s \leqslant a$ for some $a \in A\}$ of S, obviously $A \subseteq (A]$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, then $AB = \{ab \mid a \in A, b \in B\}$, $((A]] = (A]$, $(A](B] \subseteq (AB]$, $((A](B]] = (AB]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A \cap B] \neq (A] \cap (B]$ in general.

Now we define an ideal theory for a class of non-associative and non-commutative algebraic structures (ordered AG-groupoid), which is similar to the ideal theory of associative algebraic structure (ordered semigroup), but not identical.

For $\emptyset \neq A \subseteq S$. A is called an AG-subgroupoid of S if $A^2 \subseteq A$. A is called a left (resp. right) ideal of S if the following hold: (1) $SA \subseteq A$ (resp. $AS \subseteq A$). (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$. Equivalent definition: A is called a left (resp. right) ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ (resp. $AS \subseteq A$). A is called an ideal of S if A is both a left ideal and a right ideal of S. In particular, if A and B are some types of ideals of S, then $(A \cap B] = (A] \cap (B]$.

We denote by $L(a)$, $R(a)$, $I(a)$ the left ideal, the right ideal and the ideal of S, respectively generated by a. We have $L(a) = \{s \in S\colon\ s \leqslant a$ or $s \leqslant xa$ for some $x \in S\} = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup Sa \cup aS \cup (Sa)S]$.

An AG-subgroupoid A of S is called a bi-ideal of S if (1) $(AS)A \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$ (or $(A] \subseteq A$). A non-empty subset A of S is called a generalized bi-ideal of S if (1) $(AS)A \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$ (or $(A] \subseteq A$). Every bi-ideal of S is a generalized bi-ideal of S. An AG-subgroupoid A of S is called a $(1,2)$-ideal of S if (1) $(AS)A^2 \subseteq A$. (2) If $a \in A$ and $b \in S$ such that $b \leqslant a$ implies $b \in A$ (or $(A] \subseteq A$).

Example 1. Let $S = \{a, b, c, d, e\}$. Define multiplication "$\cdot$" in S as follows:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	c	d	e
c	a	e	b	c	d
d	a	d	e	b	c
e	a	c	d	e	b

and $\leqslant := \{(a, b), (b, c), (b, d), (c, e), (d, e)\}$, with figure