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Upward and quasi-upward planarity testing of embedded mixed graphs ☆

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ABSTRACT

Mixed graphs have both directed and undirected edges and have received considerable attention in the literature. We study two upward planarity testing problems for embedded mixed graphs, give some NP-hardness results, and describe Integer Linear Programming techniques to solve them. Experiments show the efficiency of our approach.

1. Introduction

1.1. Upward and quasi-upward planarity of directed graphs

An *upward drawing* of a digraph G is such that all the edges are drawn as curves monotonically increasing in the vertical direction, according to their orientation. Upward drawings are quite effective to visually convey hierarchical structures, and several cognitive experiments demonstrate that the presence of edge crossings in a drawing negatively affects its readability [31–33]. This scenario has motivated lots of research in the study of the so-called *upward planarity testing* problem, that is, the problem of deciding whether a planar digraph admits an upward drawing without edge crossings, also called an *upward planar drawing*. Figs. 1(a) and 1(b) show a planar digraph G and an upward planar drawing of G.

Bertolazzi et al. [6] proved that if a digraph G with n vertices has a fixed planar embedding, then testing whether G admits an upward planar drawing that preserves its embedding can be done in $O(n^2)$ time (Fig. 1(b) is a drawing that preserves the planar embedding of the digraph in Fig. 1(a)). On the other side, Garg and Tamassia [23] proved that the upward planarity testing problem in the variable embedding setting (i.e., over all planar embeddings of the input digraph) is NP-complete. Several polynomial-time upward planarity testing algorithms have been described in the literature for specific sub-families of planar digraphs [7,18,26,29], and exponential-time upward planarity testing algorithms for more general cases can be found in [5,13,15,24]. Upward planar drawings with additional properties, called *switch-regular*, have been also studied in [8,17].

We recall that an embedded planar digraph is upward planar only if it is acyclic and *bimodal*, i.e., for each vertex v all its incoming edges (as well as all its outgoing edges) are consecutive around v. However, acyclicity and bimodality are not

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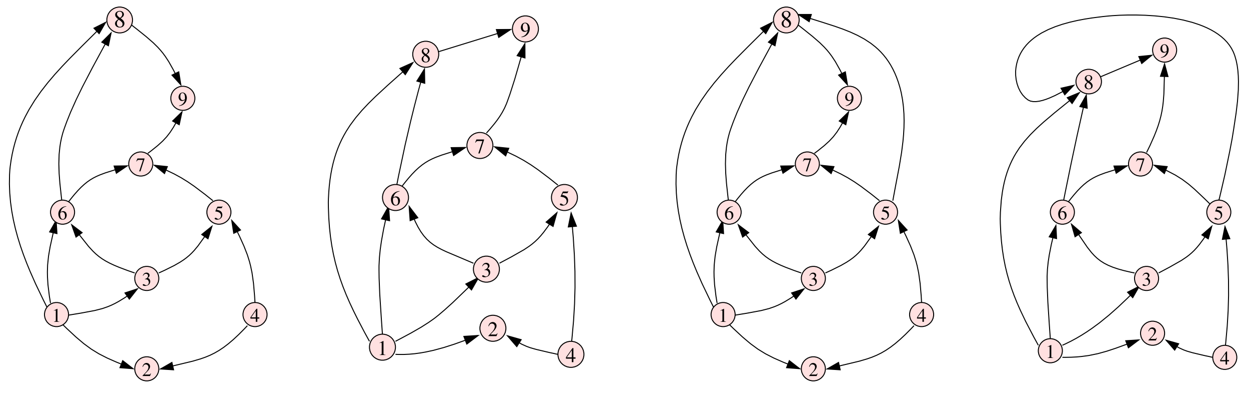

Fig. 1. (a) A planar digraph G and (b) an upward planar drawing of G. (c) A planar digraph G' that is not upward planar and (d) a quasi-upward planar drawing of G'; edge $(5, 8)$ breaks the monotonicity, but still leaves its source from above and enters its target from below.

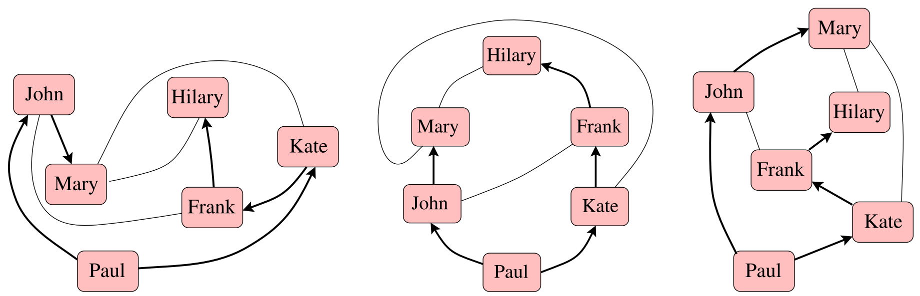

Fig. 2. (a) A planar embedded mixed graph G. (b) An embedding-preserving planar drawing of G, where the directed edges are drawn upward; the drawing is quasi-upward planar, but not upward planar. (c) An embedding-preserving planar drawing of G where the directed edges are upward and the undirected edges are vertically monotone; the drawing is upward planar.

sufficient conditions for the existence of an upward planar drawing. When a planar digraph has no upward planar drawing, relaxations of the model can be conceived. Bertolazzi et al. introduced the *quasi-upward planar* drawing convention [5]. In a quasi-upward planar drawing, directed edges are allowed to turn (i.e., to break the vertical monotonicity), but they should still enter vertices from below and leave vertices from above (see Figs. 1(c) and 1(d) for an example). In [5] Bertolazzi et al. prove that a planar embedded digraph admits a quasi-upward planar drawing if and only if it is bimodal (not necessarily acyclic), and describe a polynomial-time algorithm to compute quasi-upward planar drawings with the minimum number of edge turns (also called *bends*) when the planar embedding of the digraph is fixed.

1.2. Upward and quasi-upward planarity of mixed graphs

Many graphs arising from real applications have both directed and undirected edges. These types of graphs are called *mixed graphs* and have received considerable attention in the literature (see, e.g., [4,12,14,20,21,28]). Fig. 2(a) shows a mixed graph whose nodes represent employees of a company; the directed edges describe hierarchical relationships while the undirected edges describe collaborations. In a visual representation of a mixed graph it is still desirable that directed edges flow upward, as in Fig. 2(b). Additionally, in order to increase the readability of the layout, one may want that even the undirected edges are drawn as vertically monotone curves when possible, as in Fig. 2(c). Indeed, monotone curves have a better "geodesic tendency", which is important in comprehending the graph represented by the drawing, as confirmed by human cognitive studies [25] and by the interest in a recent visualization paradigm for undirected graphs, called *monotone drawing* [1–3].

An *upward drawing* of a mixed graph G is such that all the directed edges of G are drawn upward and all the undirected edges of G are drawn monotone in the vertical direction.

We address the following main question: *Given an embedded planar mixed graph G, does G admit an upward planar drawing that preserves its planar embedding?* The drawing in Fig. 2(c) is an embedding-preserving upward planar drawing of the graph in Fig. 2(a), while the drawing in Fig. 2(b) is not an upward drawing, because edge (Mary, Kate) is not vertically monotone (this edge has two turns). We observe that this problem is equivalent to deciding if there exists an orientation of the undirected edges of G such that the resulting embedded digraph has an upward planar drawing. If such an orientation does