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Total colorings of planar graphs with sparse triangles ☆

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ABSTRACT

The total chromatic number of a graph G, denoted by $\chi''(G)$, is the minimum number of colors needed to color the vertices and edges of G such that no two adjacent or incident elements get the same color. It is known that if a planar graph G has maximum degree $\Delta \geqslant 9$, then $\chi''(G) = \Delta + 1$. In this paper, we prove that if G is a planar graph with maximum degree 8, and for every vertex v, v is incident with at most $d(v) - 2\lfloor \frac{d(v)}{5} \rfloor$ triangles, then $\chi''(G) = 9$.

1. Introduction

All graphs considered in this paper are simple, finite, and undirected. We follow [2] for the terminology and notation not defined here. For a graph G, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$ (or simply V, E and Δ), respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^--vertex* or a *k^+-vertex* is a vertex of degree k, at most k or at least k, respectively. A *k-cycle* is a cycle of length k. A 3-cycle is also called a *triangle*. We say that two cycles are *adjacent* if they share at least one edge. We use $(v_1, v_2, \ldots, v_k)$ to denote a cycle (or a face) whose boundary vertices are consecutively $v_1, v_2, \ldots, v_k$.

A *k-total-coloring* of G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is *totally k-colorable* if it admits a k-total-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a k-total-coloring. Clearly, $\chi''(G) \geqslant \Delta + 1$. Behzad [1] and Vizing [14] conjectured independently that $\chi''(G) \leqslant \Delta + 2$ for each graph G. This conjecture was confirmed for general graphs with $\Delta \leqslant 5$. For planar graphs the only open case of this conjecture is $\Delta = 6$ [9,11]. In recent years, the study of total colorings for the class of planar graphs has attracted considerable attention. For planar graphs with large maximum degree, it is possible to determine $\chi''(G) = \Delta + 1$. This first result was given in [4] for $\Delta \geqslant 14$, which was finally extended to $\Delta \geqslant 9$ in [10]. Shen and Wang [12] conjectured that planar graphs are totally $(\Delta + 1)$-colorable, for $4 \leqslant \Delta \leqslant 8$. There are several known results to support this conjecture, see [15]. Du et al. [8] proved that planar graphs with maximum degree 8 and without adjacent triangles are totally 9-colorable. Cai et al. [7] proved that planar graphs with maximum degree 8 and without intersecting chordal 4-cycles are totally 9-colorable. Shen et al. [13] proved that planar graphs with $\Delta \geqslant 8$ and without 5- or 6-cycles with chords are totally $(\Delta + 1)$-colorable. Chang et al. [6] proved that planar graphs with $\Delta \geqslant 8$ and without 5-cycles with two chords are totally $(\Delta + 1)$-colorable. Here, we generalize these four results and get the following result.

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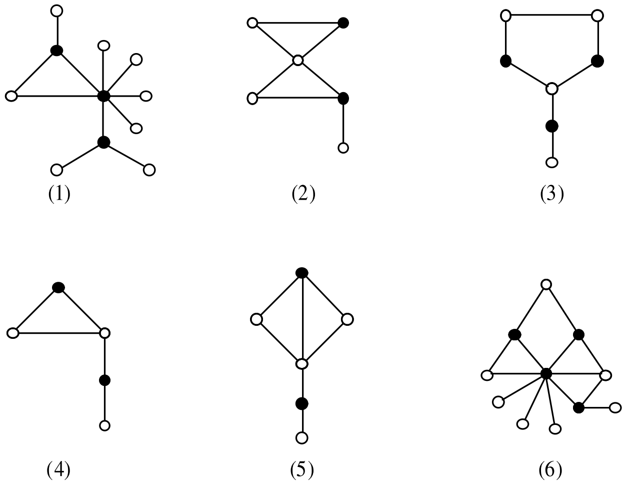

Fig. 1. Reducible configurations in G.

Theorem 1. *Let G be a planar graph with $\Delta \geqslant 8$. If every vertex v of G is incident with at most $d(v) - 2\lfloor \frac{d(v)}{5} \rfloor$ triangles, then $\chi''(G) = \Delta + 1$.*

2. Proof of Theorem 1

First, we introduce some more notations and definitions. Let G be a planar graph with a plane drawing, denote by F the face set of G. For a face f of G, the *degree* $d(f)$ is the number of edges incident with it, where each cut-edge is counted twice. A k*-face*, a k^-*-face* or a k^+*-face* is a face of degree k, at most k or at least k, respectively. Denote by $n_d(v)$ the number of d-vertices adjacent to v, by $f_d(v)$ the number of d-faces incident with v.

Now, we begin to prove Theorem 1. According to [10], the theorem is true for $\Delta \geqslant 9$. So we assume in the following that $\Delta = 8$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1, such that $|V| + |E|$ is minimum. Then every proper subgraph of G has a 9-total-coloring. Let L be the color set $\{1, 2, \dots, 9\}$ for simplicity. It is easy to prove that G is 2-connected and hence the boundary of each face f is exactly a cycle. First we give some lemmas for G.

Lemma 2. *(See [3].)* (a) *G contains no edge uv with $\min\{d(u), d(v)\} \leqslant 4$ and $d(u) + d(v) \leqslant 9$.*
(b) *G contains no even cycle $(v_1, v_2, \dots, v_{2t})$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2t-1}) = 2$.*

It follows from (a) that, the two neighbors of a 2-vertex are all 8-vertices, and any two 4^--vertices are not adjacent. Note that in all figures of the paper, vertices marked $\bullet$ have no edges of G incident with them other than those shown.

Lemma 3. *(See [6,8].) G has no configurations depicted in Fig. 1(1)–(6).*

Lemma 4. *(See [5].) Suppose that v is an 8-vertex and $v_1, v_2, \dots, v_k$ are consecutive neighbors of v with $d(v_1) = d(v_k) = 2$ and $d(v_i) \geqslant 3$ for $2 \leqslant i \leqslant k-1$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leqslant i \leqslant k-1$, then at least one vertex in $\{v_2, v_3, \dots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 5. *(See [16].) Suppose that v is an 8-vertex and $u, v_1, v_2, \dots, v_k$ are consecutive neighbors of v with $d(u) = d(v_1) = 2$ and $d(v_i) \geqslant 3$ for $2 \leqslant i \leqslant k$, where $k \in \{3, 4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $1 \leqslant i \leqslant k-2$, and the face incident with v, v_{k-1}, v_k is a 3-face, then at least one vertex in $\{v_2, v_3, \dots, v_{k-1}\}$ is a 4^+-vertex.*

Lemma 6. *(See [6].) Suppose that v is an 8-vertex and $u, v_1, v_2, \dots, v_k$ are consecutive neighbors of v with $d(u) = 2$ and $d(v_i) \geqslant 3$ for $1 \leqslant i \leqslant k$, where $k \in \{4, 5, 6, 7\}$. If the face incident with v, v_i, v_{i+1} is a 4-face for all $2 \leqslant i \leqslant k-2$, and the face incident with v, v_j, v_{j+1} is a 3-face for all $j \in \{1, k-1\}$, then at least one vertex in $\{v_2, v_3, \dots, v_{k-1}\}$ is a 4^+-vertex.*

Let φ be a (partial) 9-total-coloring of G. For a vertex $v \in V$, we denote by $C(v)$ the set of colors of edges incident with v. We say φ is *nice* if only some 4^--vertices are not colored. Note that every nice coloring can be greedily extended to a 9-total-coloring of G, since each 4^--vertex is adjacent to at most four vertices and incident with at most four edges. Therefore, in the rest of this paper, we shall always suppose that such vertices are colored at the very end.