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Total coloring of planar graphs with maximum degree 8

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ABSTRACT

Let G be a planar graph with $\Delta \geqslant 8$ and without adjacent cycles of size i and j, for some $3 \leqslant i \leqslant j \leqslant 5$. In this paper, it is proved that G is $(\Delta+1)$-total-colorable.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for terminologies and notations not defined here. Let G be a graph. We use V, E, Δ and δ to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. A k*-vertex*, k^-*-vertex* or k^+*-vertex* is a vertex of degree k, at most k or at least k, respectively. Similarly, we define a k*-face*, k^-*-face* and k^+*-face*. We say that two cycles are *adjacent* if they share at least one edge. We use $(v_1, v_2, \ldots, v_n)$ to denote a cycle whose vertices are consecutively $v_1, v_2, \ldots, v_n$.

A k*-total-coloring* of a graph $G = (V, E)$ is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. A graph G is k*-total-colorable* if it admits a k-total-coloring. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G is k-total-colorable. Clearly, $\chi''(G) \geqslant \Delta+1$. Behzad, and Vizing independently, posed the famous conjecture, known as the *Total Coloring Conjecture* (TCC).

Conjecture 1. *For any graph G, $\chi''(G) \leqslant \Delta+2$.*

Conjecture 1 was confirmed for graphs with $\Delta \leqslant 5$ [9]. In recent years, the study of total colorings of planar graphs has attracted considerable attention. For planar graphs, the only open case of Conjecture 1 is that of $\Delta = 6$ [9,13]. For graphs embedded in a surface Σ of Euler characteristic $\chi(\Sigma) \geqslant 0$, the only open case of Conjecture 1 is also that of $\Delta = 6$ [16,20]. Furthermore, the total chromatic number of planar graphs with higher maximum degree can be determined. More precisely, it is known that $\chi''(G) = \Delta+1$ if G is a planar graph with $\Delta \geqslant 9$ [10]. Some related results can be found in [3,4,6,7,11,12,17,19]. For a planar graph G with $\Delta \geqslant 8$, it is known that $\chi''(G) = \Delta+1$ if G does not contain 5-cycles or 6-cycles [8], or intersecting triangles [14,18], or adjacent triangles [5], or adjacent 4-cycles [15]. In this paper, we generalize some of the former results and get the following theorem.

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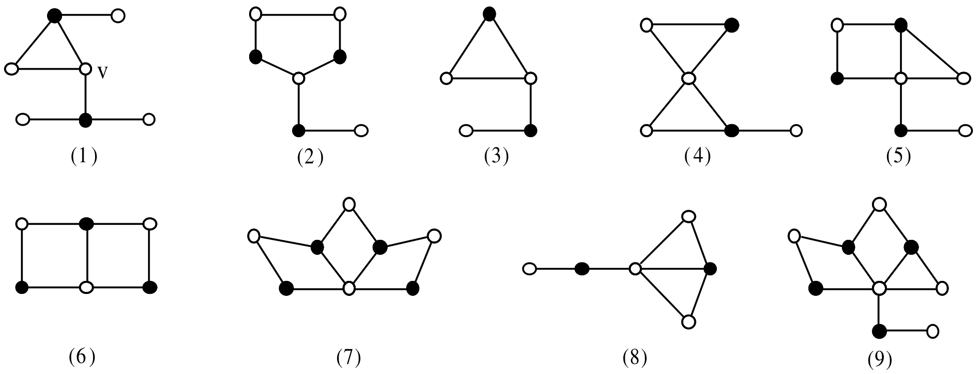

Fig. 1. Reducible configurations of Lemma 2, where $d(v) = 7$ in (1).

Theorem 1. *Suppose that G is a planar graph with $\Delta \geqslant 8$ and without adjacent cycles of size i and j, for some $3 \leqslant i \leqslant j \leqslant 5$. Then $\chi''(G) = \Delta + 1$.*

2. Reducible configurations

According to [10], planar graphs with $\Delta \geqslant 9$ are $(\Delta + 1)$-total-colorable. Therefore, to prove Theorem 1, it suffices to prove the case of $\Delta = 8$. Let $G = (V, E, F)$ be a minimal counterexample to Theorem 1, in the sense that the quantity $|V| + |E|$ is minimum. By [13], every planar graph with $\Delta \leqslant 7$ has a 9-total-coloring, so every proper subgraph of G has a 9-total-coloring φ using the color set $C = \{1, 2, \ldots, 9\}$. For a vertex v, let $C(v)$ be the set of colors assigned to the edges incident with v, and let $C'(v) = C \backslash (C(v) \cup \{\varphi(v)\})$. This section is devoted to investigating some structural information, which shows that certain configurations are *reducible*, that is, they cannot occur in G.

Lemma 1. *(See [1].)*

(1) *G is 2-connected.*
(2) *If uv is an edge of G with $d(u) \leqslant 4$, then $d(u) + d(v) \geqslant \Delta + 2 = 10$.*
(3) *The subgraph G_{28} of G induced by all edges joining 2-vertices to 8-vertices is a forest.*

In each component T of G_{28}, there is a matching M in T which pairs off all the 2-vertices with some of the 8-vertices: in the graph T, choose an 8-vertex u as the root of T, and math each 2-vertex v with the 8-vertex w adjacent to v which is farther than its another neighbor z from u (note that z is also an 8-vertex and the leaves of T are all 8-vertices). In this case, the vertex w is called the *child* of v, and the vertex z is called the *parent* of v. So every 2-vertex has exactly one parent and exactly one child, which are all 8-vertices. Moreover, if an 8-vertex is adjacent to at least two 2-vertices, then this 8-vertex is a child of at most one 2-vertex and the parent of the remaining 2-vertices adjacent to it.

Lemma 2. *(See [5,10].) G has no configurations depicted in Fig. 1(1)–(9), where the vertices marked by • have no other neighbors in G, and $d(v) = 7$ in Fig. 1(1).*

Lemma 3. *Suppose that v is an 8-vertex, u is a neighbor of v with $d(u) = 2$, and $v_1, v_2, \ldots, v_k$ are consecutive neighbors of v with $d(v_i) \geqslant 3$ for $1 \leqslant i \leqslant k$, where $k \in \{4, 5, \ldots, 7\}$. If v is incident with two 3-cycles (v_1, v, v_2) and (v_{k-1}, v, v_k), and incident with 4-cycles (v, v_i, x_i, v_{i+1}) for $2 \leqslant i \leqslant k-2$, then at least one vertex in $\{v_2, v_3, \ldots, v_{k-1}\}$ is a 4^+-vertex.*

Proof. Assume to the contrary that $d(v_i) = 3$ for $i = 2, 3, \ldots, k-1$ (see Fig. 2(1)). Let w be the neighbor of u different from v in G. By the minimality of G, $G' = G - uv$ has a 9-total-coloring φ. Erase the colors on $u, v_2, v_3, \ldots, v_{k-1}$ and without loss of generality, we assume that $\varphi(uw) = 1$. Then $1 \notin C(v) \cup \{\varphi(v)\}$, since otherwise, vu touches nine colors and can be colored properly. Without loss of generality, let $\varphi(vv_1) = 2$ and $\varphi(vv_k) = 3$. It is easy to see that $1 \in C(v_i)$ for $i = 2, 3, \ldots, k-1$, since otherwise, recolor vv_i with 1, color vu with $\varphi(vv_i)$ and color $u, v_2, v_3, \ldots, v_{k-1}$ properly, also a contradiction. We also have $1 \in \{\varphi(v_1v_2) \cup \varphi(v_{k-1}v_k)\}$, since otherwise, there is a vertex x_t $(2 \leqslant t \leqslant k-2)$ on which color 1 appears twice.

First, suppose $\varphi(v_1v_2) = \varphi(v_{k-1}v_k) = 1$. Then we have $\varphi(v_2x_2) = 2$, since otherwise, exchange the colors on vv_1 and v_1v_2, color vu with 2 and color $u, v_2, v_3, \ldots, v_{k-1}$ properly, which yields a proper coloring of G. Similarly, $\varphi(v_{k-1}x_{k-2}) = 3$. Suppose that $\varphi(v_3x_2) \neq 1$. Now, $\varphi(v_3x_3) = \varphi(v_4x_4) = \cdots = \varphi(v_{k-2}x_{k-2}) = 1$. Thus $\varphi(v_{k-2}x_{k-3}) = 3$, since otherwise, we may get a contradiction by exchanging the colors, respectively, on vv_k and $v_{k-1}v_k$, and on $v_{k-1}x_{k-2}$ and $v_{k-2}x_{k-2}$, coloring vu with 3 and coloring $u, v_2, v_3, \ldots, v_{k-1}$ properly. Similarly, $\varphi(v_{k-3}x_{k-4}) = \varphi(v_{k-4}x_{k-5}) = \cdots = \varphi(v_3x_2) = 3$. Then