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Finding good 2-partitions of digraphs II. Enumerable properties

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ABSTRACT

We continue the study, initiated in [3], of the complexity of deciding whether a given digraph D has a vertex-partition into two disjoint subdigraphs with given structural properties and given minimum cardinality. Let $\mathcal{E}$ be the following set of properties of digraphs: $\mathcal{E}$ ={strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}. In this paper we determine, for all choices of $\mathbb{P}_1, \mathbb{P}_2$ from $\mathcal{E}$ and all pairs of fixed positive integers k_1, k_2, the complexity of deciding whether a digraph has a vertex partition into two digraphs D_1, D_2 such that D_i has property $\mathbb{P}_i$ and $|V(D_i)| \geq k_i$, $i = 1, 2$. We also classify the complexity of the same problems when restricted to strongly connected digraphs. The complexity of the analogous problems when $\mathbb{P}_1 \in \mathcal{H}$ and $\mathbb{P}_2 \in \mathcal{H} \cup \mathcal{E}$, where $\mathcal{H}$ ={acyclic, complete, arc-less, oriented (no 2-cycle), semicomplete, symmetric, tournament} were completely characterized in [3].

1. Introduction

A k**-partition** of a (di)graph D is a partition of $V(D)$ into k disjoint sets. Let $\mathbb{P}_1, \mathbb{P}_2$ be two (di)graph properties, then a $(\mathbb{P}_1, \mathbb{P}_2)$**-partition** of a (di)graph D is a 2-partition (V_1, V_2) where V_1 induces a (di)graph with property $\mathbb{P}_1$ and V_2 a (di)graph with property $\mathbb{P}_2$. For example a $(\delta^+ \geq 1, \delta^+ \geq 1)$-partition is a 2-partition of a digraph where each partition induces a subdigraph with minimum out-degree at least 1.

There are many papers dealing with vertex-partition problems on (di)graphs. Examples (from a long list) are [1,3,8,9,11,12,14,15,17–21,23,24,26–28,30]. Let $\mathcal{E}$ be the following set of properties of digraphs: $\mathcal{E}$ ={strongly connected, connected, minimum out-degree at least 1, minimum in-degree at least 1, minimum semi-degree at least 1, minimum degree at least 1, having an out-branching, having an in-branching}. In [3] a systematic study of the complexity 2-partition problems

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Table 1
Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem for some properties $\mathbb{P}_1, \mathbb{P}_2$.

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc^L	NPc	P	P	P
conn.	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\mathbb{B}^+$	NPc^R	P	P	NPc	P	NPc	P	NPc	P	P	P
$\mathbb{B}^-$	NPc^R	P	NPc	P	P	P	NPc	NPc	P	P	P
$\delta \geq 1$	NPc^R	P	P	P	P	NPc	NPc	NPc	P	P	P
$\delta^+ \geq 1$	NPc^R	NPc	NPc	P	NPc	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	NPc	P	NPc	NPc	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	NPc	NPc	NPc	NPc	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

Properties: conn.: connected; $\mathbb{B}^+$: out-branchable; $\mathbb{B}^-$: in-branchable; $\mathbb{A}$: acyclic; $\mathbb{C}$: complete; $\mathbb{X}$: any property in 'being independent', 'being oriented', 'being semi-complete', 'being a tournament' and 'being symmetric'.

Complexities: P: polynomial-time solvable; NPc: NP-complete for all values of k_1, k_2; NPc^L: NP-complete for $k_1 \geq 2$, and polynomial-time solvable for $k_1 = 1$. NPc^R: NP-complete for $k_2 \geq 2$, and polynomial-time solvable for $k_2 = 1$.

Table 2
Complexity of the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem on strong digraphs.

$\mathbb{P}_1 \setminus \mathbb{P}_2$	strong	conn.	$\mathbb{B}^+$	$\mathbb{B}^-$	$\delta \geq 1$	$\delta^+ \geq 1$	$\delta^- \geq 1$	$\delta^0 \geq 1$	$\mathbb{A}$	$\mathbb{C}$	$\mathbb{X}$
strong	NPc	**P**	NPc*	NPc*	**P**	NPc^L	NPc^L	NPc	P	P	P
conn.	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\mathbb{B}^+$	NPc*	P	P	NPc*	P	NPc^L	P	NPc^L	P	P	P
$\mathbb{B}^-$	NPc*	P	NPc*	P	P	P	NPc^L	NPc^L	P	P	P
$\delta \geq 1$	**P**	P	P	P	P	**P**	**P**	**P**	P	P	P
$\delta^+ \geq 1$	NPc^R	**P**	NPc^R	P	**P**	P	NPc	NPc	P	P	P
$\delta^- \geq 1$	NPc^R	**P**	P	NPc^R	**P**	NPc	P	NPc	P	P	P
$\delta^0 \geq 1$	NPc	**P**	NPc^R	NPc^R	**P**	NPc	NPc	NPc	P	P	P
$\mathbb{A}$	P	P	P	P	P	P	P	P	NPc	P	NPc
$\mathbb{C}$	P	P	P	P	P	P	P	P	P	P	P
$\mathbb{X}$	P	P	P	P	P	P	P	P	NPc	P	P

The legend is the same as in Table 1, but we have one more complexity type: NPc*: NP-complete for $k_1, k_2 \geq 2$, and polynomial-time solvable for $k_1 = 1$ or $k_2 = 1$. We also emphasize with a bold **P**, the problems that are polynomial-time solvable on strong digraphs and NP-complete in the general case.

for digraphs was initiated and a full characterization was given for the case where one part in the 2-partition has a property from the set $\mathcal{H}$ and the other from $\mathcal{H} \cup \mathcal{E}$. See Tables 1 and 2. In this paper we provide the last entries in those tables by determining the complexity of those partition problems where both parts are required to have a given property from $\mathcal{E}$. Each of these properties $\mathbb{P}$ are **enumerable**: any given digraph D has only a polynomial number of inclusionwise maximal induced subdigraphs with property $\mathbb{P}$ and all of those can be found in polynomial time (see [3, Lemma 2.2]).

For each of the 36 distinct 2-partition problems that we study, it can be checked in linear time whether the given digraph has this property. Hence all of them are in NP. Several of these 36 $(\mathbb{P}_1, \mathbb{P}_2)$-partition problems are NP-complete and some results are somewhat surprising. For example, we show that the $(\delta^+ \geq 1, \delta \geq 1)$-partition problem is NP-complete. Some other problems are polynomial-time solvable because (under certain conditions) there are trivial $(\mathbb{P}_1, \mathbb{P}_2)$-partitions (V_1, V_2) with $|V_1| = 1$ (or $|V_2| = 1$). Therefore, in order to avoid such trivial partitions we consider **$[k_1, k_2]$-partitions**, that is, partitions (V_1, V_2) of V such that $|V_1| \geq k_1$ and $|V_2| \geq k_2$. Consequently, for each pair of above-mentioned properties and all pairs (k_1, k_2) of positive integers, we consider the $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition problem, which consists in deciding whether a given digraph D has a $(\mathbb{P}_1, \mathbb{P}_2)$-$[k_1, k_2]$-partition. The results are summarized in the upper-left 8×8 subtable of Table 1 (all other results, in grey, are proved in [3]).

The paper is organized as follows. We first introduce the necessary terminology. In Section 3 we handle the polynomial-time solvable cases. Then in Section 4 we handle the NP-complete cases and in Section 5 we investigate the impact of strong connectivity on the complexity of the partition problems. We prove that several of the NP-complete problems become polynomial-time solvable when the input is a strong digraph. We also point out that others are still NP-complete on strong digraphs. Our results are summarized in the upper 8×8 submatrix of Table 2. Finally we discuss a few other natural 2-partition problems and pose some open problems.

2. Notation and definitions

Notation follows [2]. In this paper graphs and digraphs have no parallel edges/arcs and no loops. We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set A. We use $|D|$ to denote $|V(D)|$. Given an arc $uv \in A$ we say that u **dominates** v and v is **dominated** by u. If uv or vu (or both) are arcs of D, then u and v are **adjacent**. If none of the arcs exist in D, then u and v are **non-adjacent**. The **underlying graph** of a digraph D, denoted $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple copies of the