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A dichotomy theorem for circular colouring reconfiguration ☆

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ABSTRACT

Let p and q be positive integers with $p/q \geq 2$. The "reconfiguration problem" for circular colourings asks, given two (p,q)-colourings f and g of a graph G, is it possible to transform f into g by changing the colour of one vertex at a time such that every intermediate mapping is a (p,q)-colouring? We show that this problem can be solved in polynomial time for $2 \leq p/q < 4$ and that it is PSPACE-complete for $p/q \geq 4$. This generalizes a known dichotomy theorem for reconfiguring classical graph colourings. As an application of the reconfiguration algorithm, we show that graphs with fewer than $(k-1)!/2$ cycles of length divisible by k are k-colourable.

1. Introduction

In recent years, a large body of research has emerged concerning so-called "reconfiguration" variants of combinatorial problems (see, e.g., the survey of van den Heuvel [14] and the references therein, as well as [15,16]). These problems are typically of the following form: given two solutions to a fixed combinatorial problem (e.g. two cliques of order k in a graph or two satisfying assignments of a specific 3-SAT instance) is it possible to transform one of these solutions into the other by applying a sequence of allowed modifications such that every intermediate object is also a solution to the problem?

As a specific example, for a fixed integer k and a graph G, one may ask the following: given two (proper) k-colourings f and g of G, is it possible to transform f into g by changing the colour of one vertex at a time such that every intermediate mapping is a k-colouring?[2] In the affirmative we say that f *reconfigures* to g. This problem is clearly solvable in polynomial time for $k \leq 2$. Rather surprisingly, Cereceda, van den Heuvel and Johnson [7] proved that it is also solvable in polynomial time for $k = 3$ despite the fact that determining if a graph admits a 3-colouring is NP-complete. On the other hand, Bonsma and Cereceda [3] proved that, when $k \geq 4$, the problem is PSPACE-complete. (As pointed out in [6], a similar result was

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2 Throughout the paper, the term *k-colouring* will refer to a proper k-colouring.

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proved by Jakob [17], but in his result the integer k is part of the input.) Combining these two results yields the following dichotomy theorem:

Theorem 1 *(Cereceda, van den Heuvel and Johnson [7]; Bonsma and Cereceda [3]). The reconfiguration problem for k-colourings is solvable in polynomial time for $k \leq 3$ and is* PSPACE*-complete for $k \geq 4$.*

In this paper, we study the complexity of the reconfiguration problem for circular colourings. Given a graph G and positive integers p and q with $p/q \geq 2$, a *(circular) (p,q)-colouring* of G is a mapping $f : V(G) \to \{0, \dots, p-1\}$ such that

$$\text{if } uv \in E(G), \text{then } q \leq |f(u) - f(v)| \leq p - q. \tag{1}$$

Clearly, a $(p,1)$-colouring is nothing more than a p-colouring and so (p,q)-colourings generalize classical graph colourings. Circular colourings were introduced by Vince [22], and have been studied extensively; see the survey of Zhu [25]. Analogous to that of classical graph colourings, the reconfiguration problem for circular colourings asks, given (p,q)-colourings f and g of G, whether it is possible to reconfigure f into g by recolouring one vertex at a time while maintaining (1) throughout.

Classical graph colourings and circular colourings are both special cases of graph homomorphisms. Recall, a *homomorphism* from a graph G to a graph H (also called an *H-colouring* of G) is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. The notation $f : G \to H$ indicates that f is a homomorphism from G to H. In this language, a k-colouring is simply a homomorphism to a complete graph on k vertices. A (p,q)-colouring of G is equivalent to a homomorphism from G to the graph $G_{p,q}$ which has vertex set $\{0, \dots, p-1\}$ and edge set $\{ij : q \leq |i-j| \leq p-q\}$. The graph $G_{p,q}$ is called a *circular clique*.

Remark 2. It is well known that $G_{p,q}$ admits a homomorphism to $G_{p',q'}$ if and only if $p/q \leq p'/q'$ [22]. Therefore, since the composition of two homomorphisms is a homomorphism, a graph G admits a (p,q)-colouring if and only if it admits a (p',q')-colouring for all $p'/q' \geq p/q$.

Given two homomorphisms $f, g : G \to H$, we say f *reconfigures* to g if there a sequence $(f = f_0), f_1, f_2, \dots, (f_n = g)$ of homomorphisms from G to H such that f_i and f_{i+1} differ on only one vertex. The sequence is referred to as a *reconfiguration sequence*. Clearly the existence of a reconfiguration sequence from f to g can be determined independently for each component of G, so we may assume that G is connected. We define the general homomorphism reconfiguration problem as follows. Let H be a fixed graph.

H-Recolouring
Instance: A connected graph G, and two homomorphisms $f, g : G \to H$.
Question: Does f reconfigure to g?

When $H = K_k$ or $H = G_{p,q}$ we will call the problem k-Recolouring and (p,q)-Recolouring respectively. Thus, Theorem 1 is a dichotomy theorem for k-Recolouring. Our main result is a dichotomy theorem for (p,q)-Recolouring:

Theorem 3. *Let p, q be fixed positive integers with $p/q \geq 2$. Then the (p,q)-Recolouring problem is solvable in polynomial time for $2 \leq p/q < 4$ and is* PSPACE*-complete for $p/q \geq 4$.*

The complexity of H-Recolouring is only known for a handful of families of targets. Theorem 1 is a dichotomy theorem for the family of complete graphs and Theorem 3 is a dichotomy theorem for the family of circular cliques. Recently, Wrochna [24] (see also [23]) proved that H-Recolouring is polynomial whenever H does not contain a 4-cycle. In contrast, one can observe that $G_{p,q}$ contains 4-cycles whenever $p > 2q + 1$ and so the polynomial side of Theorem 3 does not follow directly from the result of Wrochna. In a follow-up paper [5], we determine the complexity of H-Recolouring for several additional classes of graphs including, for example, odd wheels.

The rest of the paper is outlined as follows. In Section 2, we provide an explicit polynomial-time algorithm for deciding the (p,q)-Recolouring problem when $2 \leq p/q < 4$. In Section 3, we show that, when $p/q \geq 4$, the reconfiguration problem for $\lfloor p/q \rfloor$-colourings can be reduced to the reconfiguration problem for (p,q)-colourings, thereby completing the proof of Theorem 3 (via Theorem 1). We close the paper by presenting an unpublished argument of Wrochna which uses a result of [7] (on which our algorithm is based) to show that graphs with no cycle of length 0 mod 3 are 3-colourable. This result was originally proved by Chen and Saito [8]. We then generalize Wrochna's argument to show that graphs with chromatic number greater than k must contain at least $\frac{(k-1)!}{2}$ distinct cycles of length 0 mod k and conjecture a stronger bound.

2. The polynomial cases: $2 \leq p/q < 4$

We now extend the ideas of [7] to study the complexity of (p,q)-Recolouring for $2 \leq p/q < 4$. Given two 3-colourings f and g of a graph G, the algorithm in [7] consists of two phases. The first phase tests whether the so-called "fixed vertices"