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Constructing independent spanning trees for locally twisted cubes[☆]

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ABSTRACT

The independent spanning trees (ISTs) problem attempts to construct a set of pairwise independent spanning trees and it has numerous applications in networks such as data broadcasting, scattering and reliable communication protocols. The well-known ISTs conjecture, Vertex/Edge Conjecture, states that any n-connected/n-edge-connected graph has n vertex-ISTs/edge-ISTs rooted at an arbitrary vertex r. It has been shown that the Vertex Conjecture implies the Edge Conjecture. In this paper, we consider the independent spanning trees problem on the n-dimensional locally twisted cube LTQ_n. The very recent algorithm proposed by Hsieh and Tu (2009) [12] is designed to construct n edge-ISTs rooted at vertex 0 for LTQ_n. However, we find out that LTQ_n is not vertex-transitive when $n \geq 4$; therefore Hsieh and Tu's result does not solve the Edge Conjecture for LTQ_n. In this paper, we propose an algorithm for constructing n vertex-ISTs for LTQ_n; consequently, we confirm the Vertex Conjecture (and hence also the Edge Conjecture) for LTQ_n.

1. Introduction

Two spanning trees in a graph G are said to be vertex/edge independent if they are rooted at the same vertex r and for each vertex v of G, $v \neq r$, the paths from r to v in two trees are vertex/edge disjoint except the two end vertices. A set of spanning trees of G are said to be vertex/edge independent if they are pairwise vertex/edge independent. The vertex/edge independent spanning trees (ISTs) problem attempts to construct a set of pairwise vertex/edge independent spanning trees and it has has applications such as data broadcasting, scattering and reliable communication protocols. For example, a rooted spanning tree in the underlying graph of a network can be viewed as a broadcasting scheme for data communication and fault-tolerance can be achieved by sending n copies of the message along the n independent spanning trees rooted at the source node [1]. For other applications, see [3] for the multi-node broadcasting problem, [21] for one-to-all broadcasting, and [2] for n-channel graphs, reliable broadcasting and secure message distribution.

The independent spanning trees problem has been widely studied in the last two decades. Two well-known conjectures on this problem are raised by Zehavi and Itai [27]: (refer to [4] or [23] for graph terminologies)

Conjecture 1.1 (*Vertex Conjecture*)**.** *Any n-connected graph has n vertex-ISTs rooted at an arbitrary vertex r.*

Conjecture 1.2 (*Edge Conjecture*)**.** *Any n-edge-connected graph has n edge-ISTs rooted at an arbitrary vertex r.*

Zehavi and Itai [27] also raised the question: It would be interesting to show that either the Vertex Conjecture implies the Edge Conjecture, or vice versa. Later, Khuller and Schieber [16] successfully proved that the Vertex Conjecture implies the Edge Conjecture, i.e., if any n-connected graph has n vertex-ISTs, then any n-edge-connected graph has n edge-ISTs. Khuller

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Table 1
The connectivity, edge-connectivity and diameters of Q_n and its variants.

Topology	$\kappa(G)$	$\lambda(G)$	Diameter	
Q_n	n	n	n	
LTQ_n	n	n	$\lceil (n+1)/2 \rceil$ $\lceil (n+3)/2 \rceil$	if $n < 5$ if $n \geq 5$
TQ_n	n	n	$\lceil (n+1)/2 \rceil$	
MQ_n	n	n	$\lceil (n+2)/2 \rceil$ $\lceil (n+1)/2 \rceil$	in 0-MQ_n for $n \geq 4$ in 1-MQ_n for $n \geq 1$

and Schieber's proof also works for the directed graphs. For the directed case, Edmonds [7] solved the Edge Conjecture. Khuller and Schieber [16] pointed out that the Vertex Conjecture for directed graphs is the strongest conjecture since it implies all the other conjectures.

The vertex and the edge conjectures have been confirmed only for $n \leq 4$. In particular, in [15], Itai and Rodeh proposed a linear-time algorithm for constructing two edge-ISTs for a 2-edge-connected graph; they also solved the Vertex Conjecture for $n = 2$. In [27], Zehavi and Itai solved the Vertex Conjecture for $n = 3$, but they did not proposed an algorithm for constructing three vertex-ISTs. In [6], Cheriyan and Maheshwari proposed an $O(|V(G)|^2)$-time algorithm for constructing three vertex-ISTs in a 3-connected graph. In [5], Curran et al. proposed an $O(|V(G)|^3)$-time algorithm for constructing four vertex-ISTs in a 4-connected graph. When $n \geq 5$, both the vertex and the edge conjectures are still open. It has been proven that the Vertex/Edge Conjecture holds for several restricted classes of graphs or digraphs, such as planar graphs [9,10,17,18], maximal planar graphs [19], product graphs [20], chordal rings [14,24], de Bruijn and Kautz digraphs [8,11], and hypercubes [22,26]. Note that the development of algorithms for constructing vertex-ISTs tends toward pursuing two research goals: One is to design efficient construction schemes (for example, [14,17,19,24] proposed linear-time algorithms) and the other is to reduce the heights of vertex-ISTs (for example, [11,22,24] proposed the idea of height improvements).

The hypercube (Q_n) is one of the most popular interconnection network topologies due to its simple structure and ease of implementation. Several commercial machines with hypercube topology have been built and a huge amount of research work, both theoretical and practical, has been done on various aspects of the hypercube. However, it has been shown that the hypercube does not achieve the smallest possible diameter for its resources. Therefore, many variants of the hypercube have been proposed. The most well-known variants are locally twisted cubes (LTQ_n), twisted cubes (TQ_n), crossed cubes (CQ_n) and Möbius cubes (MQ_n). A concise comparison including the connectivity, edge-connectivity and diameters of Q_n and its variants is shown in Table 1. Clearly, one advantage of LTQ_n over Q_n is that the diameter of LTQ_n is only about half of that of Q_n.

Before going further, we now briefly review results of the vertex-ISTs problem for Q_n. It is well known that Q_n is n-connected. Since Q_n is a product graph, the algorithm proposed by Obokata et al. [20] can be used to construct n vertex-ISTs for Q_n. As to the construction of the height-reduced vertex-ISTs on Q_n, Tang et al. [22] modified the algorithm in [20] and proposed an $O(n2^n)$-time algorithm for constructing an optimal set (in the sense of smallest average path lengths) of n vertex-ISTs for Q_n. It was pointed out by Yang et al. [26] that the algorithms in [20,22] are designed by a recursive fashion and such a construction forbids the possibility that the algorithm could be parallelized; Yang et al. [26] therefore proposed a parallel construction for an optimal set of n vertex-ISTs for Q_n.

The purpose of this paper is to confirm the Vertex Conjecture for the n-dimensional locally twisted cube LTQ_n. The very recent algorithm proposed by Hsieh and Tu [12] is designed to construct n edge-ISTs rooted at vertex 0 for LTQ_n. However, we find out that LTQ_n is not vertex-transitive whenever $n \geq 4$ (see Section 2). Therefore, Hsieh and Tu did not solve the Edge Conjecture for LTQ_n. In this paper, we will propose an algorithm for constructing n vertex-ISTs rooted at an arbitrary vertex of LTQ_n. Therefore, we will confirm the Vertex Conjecture for LTQ_n. Since vertex-ISTs are edge-ISTs, we also confirm the Edge Conjecture for LTQ_n.

In the remaining discussion, we will simply use ISTs to denote vertex-ISTs unless otherwise specified. This paper is organized as follows. In Section 2, we give definitions and notations used in the paper. In Section 3, we present an algorithm to construct n ISTs rooted at an arbitrary vertex of LTQ_n. In Section 4, we prove the correctness of our algorithm. Concluding remarks are given in the last section.

2. Preliminaries

All graphs in this paper are simple undirected graphs. Let G be a graph with vertex set $V(G)$ and the edge set $E(G)$. Let $x, y \in V(G)$. A path from x to y is denoted as x, y-path. The *distance* between two vertices x and y, denoted by $d(x, y)$, is the length of a shortest x, y-path. Two x, y-paths P and Q are *edge-disjoint* if $E(P) \cap E(Q) = \emptyset$. Two x, y-paths P and Q are *internally vertex-disjoint* if $V(P) \cap V(Q) = \{x, y\}$. A subgraph T of G is a *spanning tree* if T is a tree and $V(T) = V(G)$. Two spanning trees T and T' of G are *vertex-independent/edge-independent* if T and T' are rooted at the same vertex, say r, and