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A linear time algorithm for metric dimension of cactus block graphs

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ABSTRACT

An undirected graph $G = (V, E)$ has metric dimension at most k if there is a vertex set $U \subseteq V$ such that $|U| \le k$ and $\forall u, v \in V$, $u \ne v$, there is a vertex $w \in U$ such that $d_G(w, u) \ne d_G(w, v)$, where $d_G(u, v)$ is the distance (the length of a shortest path in an unweighted graph) between u and v. The metric dimension of G is the smallest integer k such that G has metric dimension at most k. A cactus block graph is an undirected graph whose biconnected components are either cycles or complete graphs. We present a linear time algorithm for computing the metric dimension of cactus block graphs.

1. Introduction

An undirected graph $G = (V, E)$ has metric dimension at most k if there is a vertex set $U \subseteq V$ such that $|U| \le k$ and $\forall u, v \in V$, $u \ne v$, there is a vertex $w \in U$ such that $d_G(w, u) \ne d_G(w, v)$, where $d_G(u, v)$ is the distance (the length of a shortest path in an unweighted graph) between u and v. The metric dimension of G is the smallest integer k such that G has metric dimension at most k. The metric dimension was independently introduced by Harary and Melter [8] and Slater [20].

If for three vertices u, v, w, $d_G(w, u) \ne d_G(w, v)$, then we say that u and v are *resolved* by vertex w. If every pair of vertices is resolved by at least one vertex of a vertex set U, then U is a *resolving set* for G. The *metric dimension* of G is the size of a minimum resolving set. Such a smallest resolving set is also called a *resolving basis* for G. In certain applications, the vertices of a resolving set are also called *landmark nodes* or *anchor nodes*. This is a common naming particularly in the theory of sensor networks.

Determining the metric dimension of a graph is a problem that has an impact on multiple research fields such as chemistry [2], robotics [13], combinatorial optimization [19] and sensor networks [11]. Deciding whether a given graph G has metric dimension at most k for a given integer k is known to be NP-complete for general graphs [7], planar graphs [6], and even Gabriel unit disk graphs [11].

There are several algorithms for computing the metric dimension in polynomial time for special classes of graphs, as for example for trees [2,13], wheels [10], grid graphs [15], k-regular bipartite graphs [18], amalgamation of cycles [12] and outerplanar graphs [6]. The approximability of the metric dimension has been studied for bounded degree, dense, and general graphs in [9]. Upper and lower bounds on the metric dimension are considered in [1,3] for further classes of graphs.

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Courcelles MSOL-theory [5] cannot be applied directly to solve the metric dimension problem on clique-width or tree-width bounded graphs by dynamic programming techniques. This is because the decision version of the metric dimension for each $k \geq 2$ cannot be defined in monadic second order logic, neither of type 1 with quantifications over vertices and vertex sets nor of type 2 with quantifications over vertices, edges, vertex sets and edge sets. This follows, for example, from the observation that there is an infinite number of 1-terminal graphs mutually not replaceable with respect to metric dimension 2, see Fig. 16, which contradicts the character of MSOL graph properties, see [4].

In this paper, we introduce a concept that allows us to compute the metric dimension of a cactus graph G in linear time. An undirected graph G is a cactus graph, if every biconnected component of G is a cycle, or equivalently, if every edge belongs to at most one simple cycle. Cactus graphs are, for example, used to model electronic circuits with specific properties [16] and have recently been considered for genome comparison [17].

The main idea of our solution is based on the construction of a forest F for G with the following properties. Every edge e of forest F represents a vertex set V_e of G such that if two edges e_1 and e_2 have a common vertex then every resolving set for G contains at least one vertex of V_{e_1} or V_{e_2}. The minimum size of a resolving set can then be determined by the maximum size of a matching for F plus a certain number of additional vertices.

A block graph is a graph whose biconnected components are complete graphs. We also show that complete biconnected components can be substituted by single vertices without altering the metric dimension of the graph. In conjunction with the results for cactus graphs this yields a linear time algorithm for determining the metric dimension of cactus block graphs.

Two remarks about similar work: The paper of Iswadi et al. [12] considers only a very special case of cactus graphs without leafs and pendant vertices. The paper of Díaz et al. [6] analyzes outerplanar graphs. They show that there is a polynomial time algorithm for computing the metric dimension of outerplanar graphs with a running time of $O(n^{12})$. The results of these papers do not imply a linear time algorithm for determining the metric dimension of cactus block graphs.

This paper is organized as follows. In Section 2, we introduce some basic definitions, in Section 3, we recall how to compute the metric dimension of trees, in Section 4, we consider sun graphs, and in Section 5, we finally analyze cactus block graphs. Conclusions are given in Section 6.

2. Basic definitions

We consider *undirected graphs* $G = (V, E)$, where V is the set of *vertices* and $E \subseteq \{\{u, v\} \mid u, v \in V,\ u \neq v\}$ is the set of *undirected edges*. A graph $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. It is an *induced subgraph* of G, denoted by $G|_{V'}$, if $E' = E \cap \{\{u, v\} \mid u, v \in V'\}$. A *path of length* k between two vertices $u, v \in V$ is a sequence of $k+1$ vertices $u_1, \ldots, u_{k+1} \in V$ such that $u_1 = u$, $\{u_i, u_{i+1}\} \in E$ for $i = 1, \ldots, k$, and $u_{k+1} = v$. Graph $G = (V, E)$ is *connected* if there is a path between every pair of vertices. The *distance* $d_G(u, v)$ between two vertices u, v in a graph G is the smallest integer k such that there is a path of length k between u and v. A vertex set $U \subseteq V$ is a *resolving set* for a connected graph G if for every vertex pair $u, v \in V$, $u \neq v$, there is a vertex $w \in U$ such that $d_G(w, u) \neq d_G(w, v)$. A resolving set of minimum size is called a *resolving basis* for G. A connected graph $G = (V, E)$ has *metric dimension* k if there is a resolving basis for G of size k.

Lemma 2.1. *Let $G = (V, E)$ be a connected graph, $V_1, V_2 \subseteq V$, $V_1 \cup V_2 = V$, and $V_1 \cap V_2 = \{u\}$ for some vertex u such that every path between two vertices $u_1 \in V_1$ and $u_2 \in V_2$ contains vertex u. If V_1 contains more than the one vertex u and if V_1 is not a path with end vertex u, then every resolving set for G contains at least one vertex of $V_1 \setminus \{u\}$.*

Proof. If V_1 contains more than the one vertex u, if V_1 is not a path with end vertex u, and if G is connected then there are at least two distinct vertices $u_i, u_j \in V_1 \setminus \{u\}$ with $d_G(u_i, u) = d_G(u_j, u)$. All paths between vertices from V_1 and V_2 pass vertex u. That is, for all vertices $v \in V_2$ we have $d_G(u_i, v) = d_G(u_j, v)$ and thus every resolving set for G has to contain at least one vertex from $V_1 \setminus \{u\}$. □

A connected subgraph of G of maximal size is called a *connected component* of G. A vertex $u \in V$ is a *separation vertex* if $G|_{V \setminus \{u\}}$ (the subgraph of G induced by $V \setminus \{u\}$) has more connected components than G.

Lemma 2.2. *Every connected graph $G = (V, E)$ has a resolving basis without separation vertices.*

Proof. Let $U \subseteq V$ be a resolving basis for G, $u_s \in U$ be a separation vertex of G, and $V_1, \ldots, V_k$, $k > 1$, be the vertex sets of the connected components of $G|_{V \setminus \{u_s\}}$ (the subgraph of G induced by $V \setminus \{u_s\}$).

If a vertex set $V_i \cup \{u_s\}$, $1 \leq i \leq k$, is a path in G with end vertex u_s, then $(U \setminus \{u_s\}) \cup \{v\}$ is a resolving basis for G, where v is the other end vertex of path $V_i \cup \{u_s\}$, and thus v is no separation vertex.

Assume now that no vertex set $V_i \cup \{u_s\}$, $1 \leq i \leq k$, is a path in G with end vertex u_s. Then by Lemma 2.1, every set V_i, $1 \leq i \leq k$, contains at least one vertex of U. Let u_1, u_2 be two vertices of V such that $d_G(u_1, u_s) \neq d_G(u_2, u_s)$.

1. If both vertices u_1 and u_2 are in the same set V_i for some i, $1 \leq i \leq k$, then there has to be a vertex $v \in U \setminus (V_i \cup \{u_s\})$ such that $d_G(u_1, v) \neq d_G(u_2, v)$.