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Relationship between conditional diagnosability and 2-extra connectivity of symmetric graphs ☆

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ABSTRACT

The conditional diagnosability and the 2-extra connectivity are two important parameters to measure ability of diagnosing faulty processors and fault-tolerance in a multiprocessor system. The *conditional diagnosability* $t_c(G)$ of G is the maximum number t for which G is conditionally t-diagnosable under the comparison model, while the 2-*extra connectivity* $\kappa_2(G)$ of a graph G is the minimum number k for which there is a vertex-cut F with $|F| = k$ such that every component of $G - F$ has at least 3 vertices. A quite natural problem is what is the relationship between the maximum and the minimum problem? This paper partially answers this problem by proving $t_c(G) = \kappa_2(G)$ for a regular graph G with some acceptable conditions. As applications, the conditional diagnosability and the 2-extra connectivity are determined for some well-known classes of vertex-transitive graphs, including, star graphs, (n, k)-star graphs, alternating group networks, (n, k)-arrangement graphs, alternating group graphs, Cayley graphs obtained from transposition generating trees, bubble-sort graphs, k-ary n-cube networks, dual-cubes, pancake graphs and hierarchical hypercubes as well. Furthermore, many known results about these networks are obtained directly.

1. Introduction

Throughout this paper, unless otherwise specified, a graph $G = (V, E)$ is always assumed to be a simple and connected graph, where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set of G. We follow [41] for terminologies and notations not defined here.

Two distinct vertices x and y in G are adjacent if $xy \in E(G)$ and non-adjacent otherwise. If $xy \in E(G)$, then y (resp. x) is a neighbor of x (resp. y). The neighbor-set of x is denoted by $N_G(x) = \{y \in V(G) : xy \in E(G)\}$. For a subset $X \subset V(G)$, the notation $G - X$ denotes the subgraph obtained from G by deleting all vertices in X and all edges incident with vertices in X, and let $\overline{X} = V(G - X)$.

It is well known that a topological structure of an interconnection network N can be modeled by a graph $G = (V, E)$, where V represents the set of components such as processors and E represents the set of communication links in N (see a text-book by Xu [42]). Faults of some processors and/or communication lines in a large-scale system are inevitable. People are concerned with how to diagnose faults and to determine fault tolerance of the system.

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A vertex in a graph G is called a *fault-vertex* if it corresponds a faulty processor in the interconnection network N when it is modeled by G. A subset $F \subseteq V(G)$ is called a *fault-set* if every vertex in F is a faulty vertex in G, and is *fault-free* if it contains no faulty vertex in G. A fault-set F is called a *conditional fault-set* if $N_G(x) \nsubseteq F$ for any $x \in \overline{F}$. The pair (F_1, F_2) is called a *conditional fault-pair* if both F_1 and F_2 are conditional fault-sets.

The ability to identify all faulty processors in a multiprocessor system is known as system-level diagnosis. Several system-level self-diagnosis models have been proposed for a long time. One of the most important models is the *comparison diagnosis model*, shortly *comparison model*. Throughout this paper, we only consider the comparison model.

The comparison model was proposed by Malek and Maeng [36,37]. A node can send a message to any two of its neighbors which then send replies back to the node. On receipt of these two replies, the node compares them and proclaims that at least one of the two neighbors is faulty if the replies are different or that both neighbors are fault-free if the replies are identical. However, if the node itself is faulty then no reliance can be placed on this proclamation. According as that the two outputs are identical or different, one gets the outcome to 0 or 1. The collection of all comparison results forms a syndrome, denoted by σ.

A subset $F \subseteq V(G)$ is a *compatible fault-set* of a syndrome σ or σ is *compatible with* F, if σ can arise from the circumstance that F is a fault-set and $\overline{F}$ is fault-free. Let $\sigma_F = \{\sigma : \sigma \textit{ is} \text{ compatible with } F\}$. A pair (F_1, F_2) of two distinct compatible fault-sets is *distinguishable* if and only if $\sigma_{F_1} \cap \sigma_{F_2} = \emptyset$, and (F_1, F_2) is *indistinguishable* otherwise.

For a positive integer t, a graph G is *conditionally t-diagnosable* if every syndrome σ has a unique conditional compatible fault-set F with $|F| \leqslant t$. The *conditional diagnosability* of G under the comparison model, denoted by $t_c(G)$ and proposed by Lai et al. [29], is the maximum number t for which G is conditionally t-diagnosable. The conditional diagnosability better reflects the self-diagnostic capability of networks under more practical assumptions, and has received much attention in recent years. The diagnosability of many interconnection networks have been determined, see, for example, [2,3,13–15,19,28,40]. A survey on this field, from the earliest theoretical models to new promising applications, is referred to Duarte et al. [12].

A subset $X \subset V(G)$ is called a *vertex-cut* if $G - X$ is disconnected. A vertex-cut X is called a k-cut if $|X| = k$. The *connectivity* $\kappa(G)$ of G is defined as the minimum number k for which G has a k-cut.

Fault-tolerance or reliability of a large-scale parallel system is often measured by the connectivity $\kappa(G)$ of a corresponding graph G. However, the connectivity has an obvious deficiency because it tacitly assumes that all vertices adjacent to the same vertex of G could fail at the same time, but that is almost impossible in practical network applications. To compensate for this shortcoming, Fàbrega and Fiol [16] proposed the concept of the extra connectivity.

For a non-negative positive integer h, a vertex-cut X is called an *R_h-vertex-cut* if every component of $G - X$ has at least $h + 1$ vertices. For an arbitrary graph G, R_h-vertex-cuts do not always exist for some h. For example, a cycle of order 5 contains no R_2-vertex-cut. A graph G is called an R_h-graph if it contains at least one R_h-vertex-cut. For an R_h-graph G, the *h-extra connectivity* of G, denoted by $\kappa_h(G)$, is defined as the minimum number k for which G contains an R_h-vertex-cut F with $|F| = k$. Clearly, $\kappa_0(G) = \kappa(G)$. Thus, the h-extra connectivity is a generalization of the classical connectivity and can provide more accurate measures regarding the fault-tolerance or reliability of a large-scale parallel system and therefore, it has received much attention (see Xu [42] for details). We are interested in the 2-extra connectivity of a graph in this paper.

Clearly, for a graph G there are two problems here, one is the maximizing problem – conditional diagnosability $t_c(G)$, and another is the minimizing problem – the 2-extra connectivity $\kappa_2(G)$. A quite natural problem is what is the relationship between the maximum and the minimum problems? In the current literature, people are still determining these two problems independently for some classes of graphs, such as alternating group network [47], alternating group graph [20,45,51], the 3-ary n-cube network [46].

In this paper, we reveal the relationships between the conditional diagnosability $t_c(G)$ and the 2-extra connectivity $\kappa_2(G)$ of a regular graph G with some acceptable conditions by establishing $t_c(G) = \kappa_2(G)$. As applications of our result, we consider some more general well-known classes of vertex-transitive graphs, such as star graphs, (n, k)-star graphs, alternating group networks, (n, k)-arrangement graphs, alternating group graphs, Cayley graphs obtained from transposition generating trees, bubble-sort graphs, k-ary n-cube networks, dual-cubes and pancake graphs, and obtain the conditional diagnosability under the comparison model and the 2-extra connectivity of these graphs, which contain all known results on these graphs.

The rest of the paper is organized as follows. Section 2 first recalls some necessary notations and lemmas, then establishes the relationship between the conditional diagnosability and the 2-extra connectivity of regular graphs with some conditions. As applications of our main result, Section 3 determines the conditional diagnosability and the 2-extra connectivity for some well-known classes of vertex-transitive graphs.

2. Main results

We first recall some terminologies and notation used in this paper. Let $G = (V, E)$ be a graph, where $V = V(G)$, $E = E(G)$ and $|V(G)|$ is the order of G.

A sequence $(x_1, \ldots, x_n)$ of n $(\geqslant 3)$ distinct vertices with $x_i x_{i+1} \in E(G)$ for each $i = 1, \ldots, n-1$ is called an n-path, denoted by P_n, if $x_1 x_n \notin E(G)$, and called an n-cycle, denoted by C_n, if $x_1 x_n \in E(G)$. A cycle C in G is *chordless* if any two non-adjacent vertices of C are non-adjacent in G.