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Note

To catch a falling robber

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ABSTRACT

We consider a Cops-and-Robber game played on the subsets of an n-set. The robber starts at the full set; the cops start at the empty set. In each round, each cop moves up one level by gaining an element, and the robber moves down one level by discarding an element. The question is how many cops are needed to ensure catching the robber when the robber reaches the middle level. Alan Hill posed the problem and provided a lower bound of $2^{n/2}$ for even n and $\binom{n}{\lceil n/2\rceil}2^{-\lfloor n/2\rfloor}$ (which is asymptotic to $2^{\lceil n/2\rceil}/\sqrt{\pi n/2}$) for odd n. Until now, no nontrivial upper bound was known. In this paper, we prove an upper bound that is within a factor of $O(\ln n)$ of this lower bound.

1. Introduction

The game of *Cops-and-Robber* is a pursuit game on a graph. In the classical form, there is one robber and some number of cops. The players begin by occupying vertices, first the cops and then the robber; multiple cops may simultaneously occupy the same vertex. In each subsequent round, each cop and then the robber can move along an edge to an adjacent vertex. The cops win if at some point there is a cop occupying the same vertex as the robber. The *cop number* of a graph G, written $c(G)$, is the least number of cops that can guarantee winning (all players always know each others' positions).

The game of Cops-and-Robber was independently introduced by Quilliot [15] and by Nowakowski and Winkler [12]; both papers characterized the graphs with cop number 1. The cop number as a graph invariant was then introduced by Aigner and Fromme [1]. Analysis of the cop number is the central problem in the study of the game and often is quite challenging. The foremost open problem in the field is Meyniel's conjecture that $c(G) = O(\sqrt{n})$ for every n-vertex connected graph G (first published in [5]). This problem has a relatively long history. At present we know only that the cop number is at most $n2^{-(1+o(1))\sqrt{\log_2 n}}$ (still in $n^{1-o(1)}$) for any connected graph on n vertices. This result was obtained independently by Lu and Peng [9], Scott and Sudakov [16], and Frieze, Krivelevich, and Loh [6] using probabilistic tools. For evidence supporting Meyniel's conjecture, it is natural to check first whether random graphs provide easy counterexamples. It is known that Meyniel's conjecture passes this test for binomial random graphs [4,2,10,13] and for random d-regular graphs [14]: for connected graphs in these models, the conjecture holds asymptotically almost surely. For more background on Cops-and-Robber, see [3].

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We consider a variant of the Cops-and-Robber game on a hypercube, introduced in the thesis of Alan Hill [7]. This variant restricts the initial positions and the allowed moves. The n-dimensional hypercube is the graph with vertex set $\{0,1\}^n$ (the set of binary n-tuples) in which vertices are adjacent if and only if they differ in one coordinate. View the vertices as subsets of $\{1,\dots,n\}$, and let the *kth level* consist of the k-sets – that is, the vertices whose size as subsets is k. We view $\varnothing$ as the "bottom" of the hypercube and $\{1,\dots,n\}$ as the "top", and we say that S lies *below* T when $S \subseteq T$.

The robber starts at the full set $\{1,\dots,n\}$; the cops start at the empty set $\varnothing$. On the kth round, the cops all move from level $k-1$ to level k, and then the robber moves from level $n+1-k$ to level $n-k$. If the cops catch the robber, then they do so on round $\lceil n/2\rceil$ at level $\lceil n/2\rceil$, when they move if n is odd, and by the robber moving onto them if n is even.

Let c_n denote the minimum number of cops that can guarantee winning the game. Hill [7] provided the lower bound $2^{n/2}$ for even n and $\binom{n}{\lceil n/2\rceil}2^{-\lfloor n/2\rfloor}$ for odd n; the former bound exceeds the latter by a factor of $\Theta(\sqrt{n})$. Note that here the cops have in some sense only one chance to catch the robber, on the middle level. When the cops can chase the robber by moving both up and down, the value is much smaller, with the cop number of the n-dimensional hypercube graph being $\lceil (n+1)/2\rceil$ [11].

We begin with a proof of Hill's lower bound, since its ideas motivate our arguments. (The proof below is essentially Hill's original proof, albeit presented more concisely.) We then prove our result: an upper bound within a factor of $O(\ln n)$ of this lower bound.

Theorem 1.1. *(See [7].)* $c_n \ge \begin{cases} 2^m, & n=2m; \\ \binom{2m+1}{m+1}2^{-m}, & n=2m+1. \end{cases}$

Proof. After each move by the robber, some cops may no longer lie below the robber. Such cops are effectively eliminated from the game. We call them *evaded cops*; cops not yet evaded are *surviving cops*.

Consider the robber strategy that greedily evades as many cops as possible with each move. Deleting an element from the set at the robber's current position evades all cops whose set contains that element. On the kth round, the surviving cops sit at sets of size k, and the robber has $n-k+1$ choices of an element to delete. Since each surviving cop can be evaded in k ways, the fraction of the surviving cops that the robber can evade on this move is at least $\frac{k}{n-k+1}$.

After the first m rounds, where $m = \lfloor n/2\rfloor$, the fraction of the cops that survive is at most $\prod_{i=1}^m\left(1-\frac{i}{n-i+1}\right)$. When $n=2m$, we compute

$$\prod_{i=1}^m\left(1-\frac{i}{2m-i+1}\right) = \prod_{i=1}^m \frac{2m-2i+1}{2m-i+1} = \frac{(2m)!}{(2m)!\cdot 2^m} = 2^{-m}.$$

When $n=2m+1$, we compute

$$\prod_{i=1}^m\left(1-\frac{i}{2m-i+2}\right) = \prod_{i=1}^m \frac{2m-2i+2}{2m-i+2} = \frac{2^m m!(m+1)!}{(2m+1)!} = 2^m \Big/ \binom{2m+1}{m+1}.$$

For the cops to catch the robber, at least one surviving cop must remain after m moves; this requires at least 2^m total cops when $n=2m$ and at least $\binom{2m+1}{m+1}2^{-m}$ when $n=2m+1$. $\square$

A similarly randomized strategy for the cops should produce a good upper bound. However, it is difficult to control the deviations from expected behavior over all the cops together. Our strategy will group the play of the game into phases that enable us to give essentially the same bound on undesirable deviations in each phase.

2. The upper bound

If there are enough cops to cover the entire middle level, then the robber cannot sneak through. The size of the middle level is asymptotic to $2^n/\sqrt{\pi n/2}$. This trivial upper bound is roughly the square of the lower bound in Theorem 1.1. When n is odd, a slight improvement follows by observing that one only needs to block each $(n+1)/2$-set by reaching some $(n-1)/2$-set under it. More substantial improvements use the fact that as the robber starts to move, the family of sets needing to be protected shrinks.

Our upper bound on c_n is $O(\ln n)$ times the lower bound in Theorem 1.1. We use a randomized strategy for the cops; it may or may not succeed in capturing the robber. However, with sufficiently many cops, the strategy succeeds *asymptotically almost surely* (or *a.a.s.*), that is, with probability tending to 1 as n tends to infinity. Consequently, some deterministic strategy for the cops (in response to the moves by the robber) wins the game.

To analyze our cop strategy, we need a version of the well-known Chernoff Bound:

Theorem 2.1. *(See [8].) Let X be a random variable expressed as the sum $\sum_{i=1}^n X_i$ of independent indicator random variables $X_1,\dots,X_n$, where X_i is a Bernoulli random variable with expectation p_i (the expectations need not be equal). For $0 \le \varepsilon \le 1$,*