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Some permutations on Dyck words

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A R T I C L E I N F O

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In memory of Marcel-Paul Schützenberger (1920–1996)

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A B S T R A C T

We examine three permutations on Dyck words. The first one, α, is related to the Baker and Norine theorem on graphs, the second one, β, is the symmetry, and the third one is the composition of these two. The first two permutations are involutions and it is not difficult to compute the number of their fixed points, while the third one has cycles of different lengths. We show that the lengths of these cycles are odd numbers. This result allows us to give some information about the interplay between α and β, and a characterization of the fixed points of $\alpha \circ \beta$.

1. Introduction

Dyck words are a central object in combinatorics and formal language theory. They are enumerated by Catalan numbers as many other objects, for instance Chapter 6 of [16] points out the role of Dyck words (or Dyck paths) in enumerative combinatorics.

From the point of view of formal languages they are generated by a context free grammar, whose important properties were described in [5].

In this paper we will consider Dyck words on the alphabet $A = \{a, b\}$ to represent Dyck paths consisting of up steps, represented by the letter a, and down steps, represented by the letter b. Paths are better for the visual intuition and words are better for writing proofs, but of course they represent in different ways the same combinatorial object. For this reason, almost all the examples of the paper will be given in the language of Dyck paths, while proofs are mostly written with Dyck words.

We are interested here in transformations that do not modify the length of a word. This kind of transformations was considered by many authors (see [3,7,8,10,12–14]). This paper studies two new ones. The first one is an involution which we denote by α. It was recently introduced in [6] in the context of the sandpile (or chip-firing game) model in order to determine the rank of configurations, as defined by Baker and Norine [1], for the case of the complete graph. It relies on the so-called *Cyclic Lemma* which was obtained in [9] long time before the name Dyck path was used to speak of these

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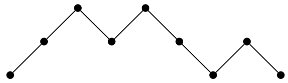

Fig. 1. A Dyck path.

combinatorial objects. This first involution is added to a very classical one, that is, the symmetry of paths along a vertical line, denoted β here, to define a new transformation γ.

The main result of the paper shows that the composition γ of the two involutions α and β has cycles of odd lengths. In order to prove this result, we remarked that it was sufficient to show that for each word w there exists an odd integer k such that $\gamma^k(w) = w$, since of course this implies that the length of the cycle of γ containing w is also odd. Then the first ingredient for the proof is to consider a subset of the set of Dyck words which we call *smooth* words, namely, Dyck words which do not have *aba* or *bab* as factors. We then associate to any Dyck word a smooth word that we call its *skeleton*. The last ingredient is to associate to each Dyck word a sequence of integers which allows to rebuild the word from its skeleton. A property of the action of γ on the skeleton and the sequence allows to end the proof.

The paper is organized as follows. In Section 2 we briefly recall the main facts about Dyck words. Section 3 contains the definition of the involutions α and β, together with a characterization of their fixed elements. In Section 4 we describe the action of the map $\gamma = \alpha \circ \beta$, while in Section 5 we define the skeleton of a Dyck word and examine its interaction with the map γ. Section 6 contains the main theorem of the paper (Theorem 3) and its consequences about the cycles of γ. Finally, in Section 7 we give a characterization of the fixed points of γ and deduce an upper bound for their numbers.

2. Dyck words and their parameters

We consider words on the alphabet with two letters $A = \{a, b\}$. For a word w, we denote $|w|_x$ (where $x \in \{a, b\}$) the number of occurrences of the letter x in w. Hence the *length* of w, denoted $|w|$, is equal to $|w|_a + |w|_b$. We use also the mapping δ associating to any word w the integer $\delta(w) = |w|_a - |w|_b$.

The word u is a *prefix* of w if $w = uv$. This prefix is *strict* if $u \neq w$.

We will use the following notation. For any word $w = w_1 w_2 \cdots w_m$ where $w_i \in A$ we denote by $\widetilde{w}$ the mirror image of w, that is, the word obtained by reading w from right to left, hence $\widetilde{w} = w_m w_{m-1} \cdots w_1$. Moreover, we denote by $\overline{w}$ the word obtained from w by replacing any occurrence of b by an occurrence of a and vice versa, giving $\overline{w} = \overline{w}_1 \overline{w}_2 \cdots \overline{w}_m$.

A Dyck word is often defined as a word w such that $\delta(w) = 0$ and $\delta(u) \geq 0$ for any prefix u of w. We consider here a slight modification adding a letter b at the end of it. We define for each non-negative integer n the set D_n of words of length $2n + 1$ first by considering A_n to be the set of words on the alphabet A having n occurrences of the letter a and $n + 1$ occurrences of the letter b. Then the set D_n is the subset of words w in A_n satisfying

$$\delta(w') \geq 0 \text{ for any strict prefix } w' \text{ of } w. \tag{1}$$

Notice that $D_0 = \{b\}$ and $D_1 = \{abb\}$.

Dyck words in D_n correspond bijectively to Dyck paths of semilength n, namely, lattice paths consisting of $2n$ steps $a = (1, 1)$ (*up steps*) and $b = (1, -1)$ (*down steps*), starting at $(0, 0)$, ending at $(0, 2n)$, and never going below the x-axis. Note that, when we see a Dyck word as a lattice path, we ignore its last letter b.

As an example, the Dyck path corresponding o the Dyck word $aababbabb$ is drawn in Fig. 1.

Many parameters are defined for Dyck words. We recall here the definition of those we use in the paper. A *peak* of a Dyck word is an occurrence of the letter a followed by an occurrence of b – or, in the Dyck path language, the vertex between an up step and a down step. Any word in D_n for $n \neq 0$ has at least one peak and at most n peaks.

The *height* of a Dyck word w is the maximum value of $\delta(u)$, as u ranges over all prefixes of w.

The number of elements of D_n are given by the Catalan numbers which satisfy the formula

$$C_n = \frac{(2n)!}{n!(n+1)!}. \tag{2}$$

An elegant way to obtain the above formula is to use the so-called *Cyclic Lemma* (see e.g. [9]) which will be used very often in this paper. This Lemma considers the set A_n and uses the operation of conjugation. Two words w and w' on an alphabet A are *conjugate* if there exist u and v such that $w = uv$ and $w' = vu$. In other terms, if you write a word w in counterclockwise order, wrapped around a circle, the conjugates of w are then all words obtained by starting reading at any letter in counterclockwise order and making a full circle.

Lemma 1. ***(Cyclic Lemma)*** *Any word w of A_n has exactly one decomposition into two factors $w = uv$ such that vu is an element of D_n. Moreover the decomposition into two factors $w = uv$ is such that u is the smallest prefix of w attaining the minimal value for $\delta(u)$.*

It is well known that the number of different conjugates of a word w divides its length and that if a word uv is equal to one of its conjugates vu, where $u, v \neq \emptyset$, then there exist a word q and an integer $k > 1$ such that $uv = q^k$ (see [15,