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Direct solution of piecewise linear systems

Manuel Radons

Department of Mathematics, Humboldt-University of Berlin, Germany

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ABSTRACT

Let S be a real $n \times n$ matrix, $z, \hat{c} \in \mathbb{R}^n$, and $|z|$ the componentwise modulus of z. Then the piecewise linear equation system

$$z - S|z| = \hat{c}$$

is called an *absolute value equation* (AVE). It has been proven to be equivalent to the general *linear complementarity problem*, which means that it is NP-hard in general.
We will show that for several system classes (in the sense of structural impositions on S) the AVE essentially retains the good-natured solvability properties of regular linear systems. I.e., it can be solved directly by a slightly modified Gaussian elimination that we call the signed Gaussian elimination. For dense matrices S this algorithm has, up to a term in $\mathcal{O}(n)$, the same operations count as the classical Gaussian elimination with column pivoting. For tridiagonal systems in n variables its computational cost is roughly that of sorting n floating point numbers. The sharpness of the proposed restrictions on S will be established.

1. Introduction and notation

We denote by $\mathrm{M}_n(\mathbb{R})$ the space of $n \times n$ real matrices, and by $[n]$ the set $\{1, \ldots, n\}$. For vectors and matrices *absolute values and comparisons are used entrywise*. Zero vectors and matrices are denoted by $\mathbf{0}$.

A *signature matrix* Σ, or, briefly, *a signature*, is a diagonal matrix with entries $+1$ or -1. The set of n-dimensional signature matrices is denoted by $\mathrm{diag}_{n,\sigma}$. A single diagonal entry of a signature is a sign σ_i $(i \in [n])$.

Let $S \in \mathrm{M}_n(\mathbb{R})$, $z, \hat{c} \in \mathbb{R}^n$. The *piecewise linear equation system* (PLE)

$$z - S|z| = \hat{c} \tag{1}$$

is called an *absolute value equation* (AVE). It was first introduced by Rohn in [15]. Mangasarian proved its equivalence to the general *linear complementarity problem* (LCP) [12]. In [13, P. 216–230] Neumaier authored a detailed survey about its intimate connection to the research field of *linear interval equations*. A recent result by Griewank and Streubel has shown that PLEs of arbitrary structure can be, with a *one-to-one solution correspondence*, transformed into an AVE [4, Lem. 6.5].

An especially closely related system type are equilibrium problems of the form

$$Ax + \max(0, x) = b, \tag{2}$$

where $A \in \mathrm{M}_n(\mathbb{R})$ and $x, b \in \mathbb{R}^n$. (A prominent example is the first hydrodynamic model presented in [1].) Using the identity $\max(s, t) = (s + t + |s - t|)/2$, equality (2) can be reformulated as

E-mail address: radons@math.hu-berlin.de.

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$$Ax + \frac{x + |x|}{2} = b \quad \Longleftrightarrow \quad (2A + I)x + |x| \equiv Bx + |x| = 2b. \tag{3}$$

For regular B, system (3) is clearly equivalent to (1).

This position at the crossroads of several interesting problem areas gives relevance to the task of developing efficient solvers for the AVE. The latest publications on the matter include approaches by linear programming [11] and concave minimization [9], as well as a variety of Newton and fixed point methods (see, e.g., [1,19,5] or [4]).

Let $\Sigma \in \mathrm{diag}_{n,\sigma}$ s.t. $\Sigma z = |z|$. (Note that, since $0 = +0 = -0$, we need no "0"-sign.) Then we can rewrite (1) as

$$(I - S\Sigma)z = \hat{c}. \tag{4}$$

In this form it becomes apparent that the main difficulty in the computation of a solution for (4) is to determine the proper signature Σ for z. That is, to determine in which of the 2^n orthants around the origin z lies. This is NP-hard in general [10].

It was proven by Rump in [16, Cor. 2.9] that checking the system for unique solvability is NP-hard as well, as it is equivalent to checking whether a quantity called the *sign-real spectral radius* of S is smaller than one, which in turn is equivalent to checking whether the system matrix of the equivalent LCP is a *P-matrix*. As these notions and results are fundamental to the understanding of the AVE, we will give a short account of them in the second section. There we will also see that the systems investigated in the present paper, which all have $\|S\|_\infty < 1$, are uniquely solvable.

The following simple observation is key to the subsequent discussion:

Proposition 1.1. *If $\|S\|_\infty < 1$, then for at least one $i \in [n]$ the signs of z_i and $\hat{c}_i$ have to coincide.*

Proof. Let z_i be an entry of z s.t. $|z_i| \geq |z_j|$ for all $j \in [n]$. If $z_i = 0$, then $z = \mathbf{0}$ and thus $\hat{c} \equiv z - S|z|$ is the zero vector as well—and the statement holds trivially. If $|z_i| > 0$, then $\left|e_i^T S|z|\right| < |z_i|$, due to the norm constraint on S. Thus, $\hat{c}_i = z_i - e_i^T S|z|$ will adopt the sign of z_i. □

We do not know though, for which indices the signs coincide. In the third section we will derive several types of structural restrictions on S, each of which will guarantee the coincidence of the signs of z_i and $\hat{c}_i$ for all $i \in [n]$ with $|\hat{c}_i|$ maximal in $\hat{c}$.

In the fourth paragraph we will devise a modified Gaussian elimination that exploits this knowledge. This *signed Gaussian elimination* (SGE) will base on the following central points:

- We are enabled to perform one step of Gaussian elimination on the AVE in the form (4), if we know the correct sign of z_1.
- If $\|S\|_\infty < 1$, no row or column pivot leads to stability or computability issues during the performance of a Gaussian elimination step on system (4). Hence, we can always produce a constellation, where $|\hat{c}_1|$ is maximal in $\hat{c}$.
- The impositions on S developed in the third paragraph are invariant under Gaussian elimination steps.

For S that conform to the restrictions derived in paragraph three, the first two points mean that we can always perform one Gaussian elimination step on (4). The third point ensures that we can repeat the procedure for the reduced system(s) and ultimately calculate the correct (unique) solution of the AVE.

We will briefly analyze the modified algorithm's runtime in the dense and tridiagonal case. For a dense matrix S the SGE solves (4) with an $\mathcal{O}(n)$ computational extra cost in comparison to the solution of a dense linear system by the Gaussian elimination with column pivoting. The additional linear term has a small constant, which makes it insignificant in comparison to the $\mathcal{O}(n^3)$ overall complexity of both the modified and unmodified algorithm. For the tridiagonal SGE the supplementary operations cost roughly as much as sorting $\hat{c}$ with respect to the absolute value of its entries. As the underlying tridiagonal Gaussian elimination, also known as the Thomas Algorithm, is in $\mathcal{O}(n)$, this means that the asymptotical complexity of the modified algorithm depends one-to-one on the implementation of the extra effort.

The paper is concluded by a discussion of the sharpness of the proposed restrictions on S.

For readers primarily interested in the algorithmic results, we remark that inequality (7), equivalence 1. $\Leftrightarrow$ 3. from Theorem 2.1, and the statements of Theorem 3.1 present the most basic preknowledge that should enable them to work with the fourth paragraph.

Note that we already outlined the approach described above in [4, Parag. 7]. This paper presents the announced elaboration on the concept.

2. Sign-real spectral radius

Denote by $\rho(S)$ the spectral radius of S and let

$$\rho_0(S) \equiv \{|\lambda| : \lambda \textit{ real} \text{ eigenvalue of } S\}$$

be the *real spectral radius* of S. Then its sign-real spectral radius is defined as follows (see [16, Def. 1.1]):