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Complexity of the maximum leaf spanning tree problem on planar and regular graphs

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ABSTRACT

In this paper, we consider the problem of finding a spanning tree in a graph that maximizes the number of leaves. We show the $\mathcal{NP}$-hardness of this problem for graphs that are planar and cubic. Our proof will be an adaption of the proof for arbitrary cubic graphs in Lemke (1988) [9]. Furthermore, it is shown that the problem is $\mathcal{APX}$-hard on 5-regular graphs. Finally, we extend our proof to k-regular graphs for odd $k > 5$.

1. Introduction

In the Maximum Leaf Spanning Tree Problem, Mlst for short, one seeks for a spanning tree in an undirected and unweighted graph that maximizes the number of leaves over all spanning trees. This problem is not only known to be $\mathcal{NP}$-complete for general graphs, ND2 in [8], but also for a range of special classes of graphs. Among these classes are planar graphs with maximum degree 4, 4-regular graphs, both [8], as well as cubic graphs, Lemke [9].

The Mlst is max $\mathcal{SNP}$-hard on general graphs, Galbiati et al. [7]. Hence, the approximability has been studied exhaustively in recent years. Particularly, Lu and Ravi gave a 3-approximation for general graphs [13] and Solis-Oba provided a 2-approximation [15]. Loryś and Zwoźniak studied the problem restricted to cubic graphs and presented a 7/4-approximation [12]. This factor has been improved to 5/3 by Correa et al. [5] and to 3/2 by Bonsma and Zickfeld [3].

Mlst is closely related to the problem of finding a *minimum connected dominating set* in a given graph $G = (V, E)$, i.e. a minimum subset $S \subseteq V$ for which $G[S]$ is connected and every vertex in $V \setminus S$ is adjacent to a vertex in S, cf. GT2 in [8]. This problem and some of its variants have many real life applications, e.g. for asymmetric multihop wireless networks [10], for static ad hoc wireless networks [11], and for the regenerator location problem in the context of optical network design [4].

In this paper, we prove the $\mathcal{NP}$-hardness of Mlst for graphs that are both, planar *and* cubic. For this, we adapt the proof of Lemke [9] for planar graphs.

In [5], the authors conjectured Mlst to be max $\mathcal{SNP}$-hard on cubic graphs. A proof of $\mathcal{APX}$-hardness of this problem was recently given in [2]. The result was obtained by a reduction from Cubic Maximum Independent Set. As stated there, the construction is simple, but needs "an elaborate global analysis of the constructed graph". We give a reduction from Minimum Dominating Set to prove $\mathcal{APX}$-hardness on 5-regular graphs. Our reduction, which is similar to the one given

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in [7], consists of a simple construction *and* an easy analysis. Remark that the MLST is $\mathcal{APX}$-hard on 5-regular graphs although the problem seems to be easier to handle on graphs with a higher regularity than 3. So it is much more involved to construct a 3/2-approximation for cubic graphs than in the case of 5-regular graphs, cf. [2].

The max $\mathcal{SNP}$-hardness proof in [7] can easily be extended to graphs with bounded degree. We go a step further and show that MLST is $\mathcal{APX}$-complete even on k-regular graphs for any odd $k \geq 5$. The results in this paper are part of the PhD thesis [14].

The paper is structured as follows. In Section 2, we fix our notations and describe the considered problems. In Section 3, we present the main components of our construction and specify the transformation for the $\mathcal{NP}$-completeness proof for MAXIMUM LEAF SPANNING TREE on planar and cubic graphs. Section 4 deals with the $\mathcal{APX}$-completeness proof for the MLST on 5-regular graphs which is done by L-reducing MINIMUM DOMINATING SET for cubic graphs. This result is extended to k-regular graphs with any odd $k > 5$ in Appendix A.

2. Preliminaries

2.1. Basic notations

Throughout the paper, we consider simple undirected graphs $G = (V, E)$ with finite node set $V(G) = V$ and finite edge set $E(G) = E$. The *degree* of a node v in G is denoted by $\deg_G(v)$. For a spanning tree T of a graph G, the number of leaves of T is denoted by $\ell(G, T)$. A graph is *k-regular* if $\deg_G(v) = k$ holds for all $v \in V$. A 3-regular graph is also called *cubic*. A graph is *planar* if it can be drawn in the plane without crossing edges. A graph that is embedded into the plane divides the plane into *faces*.

For the $\mathcal{APX}$-hardness proof we have to fix our notations of *optimization problems* and of *L-reduction*. Hereby, we mainly follow [7].

Definition 1 *(Optimization problem).* An *optimization problem* P is a 4-tuple $(\mathcal{I}, S, c, \text{opt})$, where $\mathcal{I}$ is the set of *instances* of P and $S(I)$ is the set of *feasible solutions* of an instance $I \in \mathcal{I}$. The function $c : \mathcal{I} \times S \to \mathbb{N}$ is called the *objective function*, and *opt* $\in \{\max, \min\}$. Solving P on an instance $I \in \mathcal{I}$ means to find a solution $s \in S(I)$ which maximizes respectively minimizes $c(I, s)$. We may abbreviate $\text{opt}(I) = \text{opt}\{c(I, s) \mid s \in S(I)\}$.

Definition 2 *(L-reduction).* An *L-reduction* from an optimization problem $P = (\mathcal{I}_P, S_P, c_P, \text{opt}_P)$ to an optimization problem $Q = (\mathcal{I}_Q, S_Q, c_Q, \text{opt}_Q)$ is a 4-tuple (f, g, α, β), where $f : \mathcal{I}_P \to \mathcal{I}_Q$ and $g : \mathcal{I}_P \times S_Q \to S_P$ are polynomial time computable functions, α and β are positive constants, and the following properties hold for all $I_P \in \mathcal{I}_P$ and for all $s_Q \in S_Q(I_Q)$ with $I_Q = f(I_P) \in \mathcal{I}_Q$:

1. $S_P(I_P) \neq \emptyset \Rightarrow S_Q(I_Q) \neq \emptyset$,
2. $g(I_P, s_Q) \in S_P(I_P)$,
3. $\text{opt}_Q(I_Q) \leq \alpha \cdot \text{opt}_P(I_P)$,
4. $|\text{opt}_P(I_P) - c_P(I_P, g(I_P, s_Q))| \leq \beta \cdot |\text{opt}_Q(I_Q) - c_Q(I_Q, s_Q)|$.

More informally, function f transforms an instance I_P of the problem P to an instance $I_Q = f(I_P)$ of problem Q. Now assume that there was a polynomial time approximation scheme which computes a solution $s_Q \in S_Q(f(I_P))$ for Q. Then, function g could be used to compute a solution $s_P \in S_P(I_P)$ for I_P from this transformed instance and its solution. A PTAS for problem P could then be composed by concatenating function f, the PTAS for Q, and function g. We remark that it is sufficient to define g only for instances of Q which actually arise as transformation of f. This is why we chose $\mathcal{I}_P$ as the first part of the domain of g instead of $\mathcal{I}_Q$.

2.2. The problems

MLST for cubic planar graphs is defined as follows.

3-P-MLST	
Instance:	Planar cubic graph $G = (V, E)$, positive integer $k \leq \|V\|$.
Question:	Does G have a spanning tree with at least k leaves?

Similarly to [9], we prove $\mathcal{NP}$-hardness of the more specific question about the existence of a spanning tree T on a given planar and cubic graph G_O such that there is no vertex v with $\deg_T(v) = 2$. We call this problem the PLANAR ODD DEGREE SPANNING TREE Problem, 3-P-ODST for short, when the problem is restricted to cubic graphs. ODST denotes the *odd degree spanning tree* itself. Actually, the 3-P-ODST is a special case of the 3-P-MLST with $k = \frac{n}{2} + 1$, since if there is an odd degree spanning tree in a cubic graph, it has $\frac{n}{2} - 1$ vertices of degree 3 and $\frac{n}{2} + 1$ leaves. This is also true for cubic graphs