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Space lower bounds for low-stretch greedy embeddings [☆]

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ABSTRACT

Greedy (geometric) routing is an important paradigm for routing in communication networks. It uses an embedding of the nodes of a network into points of a space (e.g., $\mathbb{R}^d$) equipped with a distance function (e.g., the Euclidean distance ℓ_2) and uses as address of each node the coordinates of the corresponding point. The embedding has particular properties so that the routing decision for a packet is taken greedily based only on the addresses of the current node, its neighbors, and the destination node and the distances of the corresponding points. In this way, there is no need to keep extensive routing information (e.g., routing tables) at the nodes. Embeddings that allow for this functionality are called greedy embeddings. Even though greedy embeddings do exist for several spaces and distance functions, they usually result in paths of high stretch, i.e., significantly longer than the corresponding shortest paths.

In this paper, we show that greedy embeddings in low-dimensional (Euclidean) spaces necessarily have high stretch. In particular, greedy embeddings of n-node graphs with optimal stretch require at least $\Omega(n)$ dimensions for distance ℓ_2. This result disproves a conjecture by Maymounkov (2006) [14] stating that greedy embeddings of optimal stretch are possible in Euclidean spaces with polylogarithmic number of dimensions. Furthermore, we present trade-offs between the stretch and the number of dimensions of the host space. Our results imply that every greedy embedding into a space with polylogarithmic number of dimensions (and Euclidean distance) has stretch $\Omega\left(\frac{\log n}{\log\log n}\right)$. We extend this result for a distance function used by an $O(\log n)$-stretch greedy embedding presented by Flury et al. (2009) [9]. Our lower bound implies that their embedding has almost best possible stretch.

1. Introduction

Greedy routing utilizes a particular assignment of node addresses so that routing of packets can be performed using only the address of the current node of a traveling packet, the addresses of its neighbors, and the address of the destination node. The node addresses are usually defined using a *greedy embedding*. The definition of a greedy embedding involves a *space of points* X and a function dist that returns a non-negative *distance* between any pair of points in X. The d-dimensional

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space $\mathbb{R}^d$, equipped with the ℓ_p distance function, is a typical example of such a space and is the focus of the current paper. The ℓ_p distance function (also known as Euclidean distance when $p = 2$) is defined as

$$\ell_p(s,t) = \begin{cases} \left(\sum_{i=1}^{d} |s_i - t_i|^p\right)^{1/p} & \text{for finite } p \geq 1 \\ \max_{i=1}^{d} |s_i - t_i| & \text{for } p = \infty \end{cases}$$

for every pair of points $s, t \in \mathbb{R}^d$. Other distance functions can also be used in greedy embeddings. Note that, in general, dist does not have to satisfy the triangle inequality; the only restriction we impose on it (in addition to non-negativity) is that $\mathrm{dist}(x,x) = 0$, for every point $x \in X$. In the following, we refer to the pair (X, dist) as the *host space*.

Formally, a greedy embedding of a graph $G = (V, E)$ into a host space (X, dist) is a function $f : V \to X$ so that the following property holds:

> for any two nodes u, t of G, there exists a node v in the neighborhood $\Gamma(u)$ of u in G so that $\mathrm{dist}(f(v), f(t)) < \mathrm{dist}(f(u), f(t))$.

Given a greedy embedding f, the coordinates of point $f(u)$ can be used as the address of node u. Then, when node u has to take a decision about the next hop for a packet with destination address $f(t)$, it has to select among its neighbors a node with address $f(v)$ such that $\mathrm{dist}(f(v), f(t)) < \mathrm{dist}(f(u), f(t))$. It is clear that, in this way, the packet is guaranteed to reach its destination within a finite number of steps.

Greedy embeddings were first defined by Papadimitriou and Ratajczak [16]. They proved that any 3-connected planar graph can be greedily embedded into $\mathbb{R}^3$ using a non-Euclidean distance function. They also conjectured that every such graph can be greedily embedded in the Euclidean plane (i.e., in $\mathbb{R}^2$ equipped with the Euclidean distance function). The conjecture was proved by Moitra and Leighton [15]. Kleinberg [11] showed that any tree can be greedily embedded in the 2-dimensional hyperbolic space. This immediately yields a greedy embedding for any graph (by just embedding a spanning tree of the graph). Eppstein and Woodrich [7] observed that the coordinates of the nodes in Kleinberg's embedding require too much space and modified it so that each coordinate is represented with $O(\log n)$ bits (where n is the size of the graph). Greedy embeddings into $O(\log n)$-dimensional spaces (with ℓ_∞ distance) are also known [11] and exploit an isometric embedding of trees due to Linial et al. [12]. An important property of this embedding is that the points of the space used have integer coordinates in $[-n, n]$. Hence, polylogarithmic space is enough to represent each point. Maymounkov [14] presents a greedy embedding of trees (and, consequently, of any graph) into $O(\log n)$-dimensional Euclidean spaces.

Note that the approach of computing the greedy embedding of a spanning tree ignores several links of the network. Hence, it may be the case that even though a packet could potentially reach its destination with a few hops using a shortest path, it is greedily routed through a path that has to travel across a constant fraction of the nodes of the whole network. The measure that can quantify this inefficiency is the *stretch* of a greedy routing algorithm, i.e., the ratio between the length of the path used by the algorithm (informally, the greedy path) over the length of the corresponding shortest path. Let us proceed with formal definitions of these terms.

Definition 1. Let f be a greedy embedding of a graph G into a space (X, dist). A path $\langle u_0, u_1, \ldots, u_t\rangle$ is called a greedy path if $u_{i+1} \in \mathrm{argmin}_{v \in \Gamma(u_i)}\{\mathrm{dist}(f(v), f(u_t))\}$, where $\Gamma(u_i)$ denotes the neighborhood of u_i in G. We say that f has stretch ρ for graph G if, for every pair of nodes u, v of G, the length of every greedy path from u to v is at most ρ times the length of the shortest path from u to v in G. The stretch of f is simply the maximum stretch over all graphs.

We use the terms *no-stretch* and *optimal stretch* to refer to embeddings with stretch equal to 1 (i.e., when a greedy path is always a shortest path).

Maymounkov [14] considers the question of whether no-stretch greedy embeddings into low-dimensional spaces exist. Among other results, he presents a lower bound of $\Omega(\log n)$ on the dimension of the host hyperbolic space for greedy embeddings with optimal stretch. Furthermore, he conjectures that any graph can be embedded into Euclidean or hyperbolic spaces with a polylogarithmic number of dimensions with no stretch. We remark that a proof of this conjecture would probably justify greedy routing as a compelling alternative to compact routing [17].

Flury et al. [9] present a greedy embedding of any n-node graph into an $O(\log^2 n)$-dimensional space that has stretch $O(\log n)$. Each coordinate in their embedding uses $O(\log n)$ bits. They used the min–max$_c$ distance function which views the d-dimensional space as composed by d/c c-dimensional spaces and, for a pair of points x, y, takes the ℓ_∞ norm of the projections of x and y into those spaces, and finally takes the minimum of those ℓ_∞ distances as the min–max$_c$ distance between them. Note that min–max$_c$ (as well as the distance function in the main result of [16]) does not satisfy the triangle inequality. The greedy embedding of [9] uses an algorithm of Awerbuch and Peleg [4] to compute a tree cover of the graph and the algorithm of Linial et al. [12] to embed each tree in the cover isometrically in a low-dimensional space, equipped with the ℓ_∞ distance. Recent experimental work on realistic scenarios has demonstrated that greedy embeddings could be useful in sensor [9] and Internet-like networks [6].

In this paper, we present lower bounds on the number of dimensions required for low-stretch greedy embeddings into Euclidean spaces. We first disprove Maymounkov's conjecture by showing that greedy embeddings into $(\mathbb{R}^d, \ell_2)$ have optimal stretch only if the number of dimensions d is linear in the graph size n. The proof uses an extension of the hard