Theoretical Computer Science 609 (2016) 104–117

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Computational complexity of covering three-vertex multigraphs

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ARTICLE INFO

Article history:
Received 26 March 2015
Received in revised form 10 August 2015
Accepted 11 September 2015
Available online 16 September 2015
Communicated by P. Widmayer

Keywords:
Computational complexity
Graph homomorphism
Covering projection

ABSTRACT

A covering projection from a graph G onto a graph H is a mapping of the vertices of G onto the vertices of H such that, for every vertex v of G, the neighborhood of v is mapped bijectively onto the neighborhood of its image. Moreover, if G and H are multigraphs, then this local bijection has to preserve multiplicities of the neighbors as well. The notion of covering projection stems from topology, but has found applications in areas such as the theory of local computation and construction of highly symmetric graphs. It provides a restrictive variant of the constraint satisfaction problem with additional symmetry constraints on the behavior of the homomorphisms of the structures involved.

We investigate the computational complexity of the problem of deciding the existence of a covering projection from an input graph G to a fixed target graph H. Among other partial results this problem has been shown NP-hard for simple regular graphs H of valency greater than 2, and a full characterization of computational complexity has been shown for target multigraphs with 2 vertices. We extend the previously known results to the ternary case, i.e., we give a full characterization of the computational complexity in the case of multigraphs with 3 vertices. We show that even in this case a P/NP-completeness dichotomy holds.

1. Introduction

The concept of covering spaces or covering projections stems from topology, but has attracted a lot of attention in algebra, combinatorics, and also the theory of computation. For instance, it is used in algebraic graph theory as a very useful tool for construction of highly symmetric graphs. The applications in computability include the theory of local computations (cf. [2] and [6]). A lot of interest has been paid to graphs that allow finite planar covers. This class of graphs is closed in the minor order and hence recognizable in polynomial time, yet in spite of a significant amount of effort no concrete recognition algorithm is known, since the obstruction set has not been determined yet. The class has been conjectured equal to projective planar graphs by Negami [16] (for the most recent results cf. [10,11]).

Deciding the existence of a covering projection between input graphs G and H was shown NP-complete by Bodlaender in 1989 [3]. In [1], Abello et al. asked to characterize the computational complexity of deciding the existence of a covering

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[1] Supported by grant CE-ITI GAČR P202/12/G061.
[2] Supported by the Norwegian Research Council, project PARALGO.
[3] Supported by Charles University by the grant SVV-2013-267313.

http://dx.doi.org/10.1016/j.tcs.2015.09.013
0304-3975/

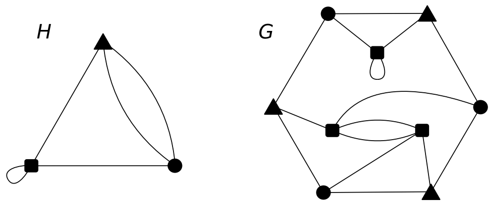

Fig. 1. Example of graph covering.

projection from an input graph G onto a fixed graph H (hoping for a P/NP-completeness dichotomy depending on H). Such a characterization seems to be hard to obtain and only very partial results are known. The most general NP-completeness result states that for every simple regular graph H of valency at least 3, the problem is NP-complete [15]. No plausible conjecture on the borderline between polynomially solvable and NP-complete instances has been published so far, yet it is believed that a P/NP-completeness dichotomy will hold similarly as in the case of the Constraint Satisfaction Problem (CSP). See [9] for a good overview of results up to 2008. More recent results include a focus on algebraically restricted coverings by planar graphs [8].

The relation to CSP is worth mentioning in more detail. As shown in [9], for every fixed graph H, the H-Cover problem can be reduced to CSP, but mostly to NP-complete cases of CSP, so this reduction does not help. In a sense a covering projection is itself a variant of CSP, but with further constraints of local symmetry. Thus the dichotomy conjecture for H-Cover does not follow from the well known Feder–Vardi dichotomy conjecture for CSP (cf. [7]).

In [14] the authors showed that in order to fully understand the H-Cover problem for simple graphs, one has to understand its generalization for colored mixed multigraphs. For this reason we are dealing with multigraphs (undirected) in this paper. Kratochvil et al. [14] completely characterized the computational complexity of the H-Cover problem for colored mixed multigraphs on two vertices. The aim of this paper is to extend this characterization to 3-vertex multigraphs (in the undirected and monochromatic case). By the results of [14] this will settle the complexity of H-Cover for infinitely many simple graphs H, e.g. those having only three vertices of degree larger than two, connected by paths all of the same length k. The characterization is described in the next section. It is more involved than the case of 2-vertex multigraphs, but this should not be surprising. Ternary structures tend to be substantially more difficult than their binary counterparts. A parallel in CSP is the dichotomy of binary CSP proved by Schaefer in the 70's [17] followed by the characterization of CSP into ternary structures by Bulatov almost 30 years later [4].

2. Preliminaries and statement of our results

For the sake of brevity we reserve the term "graph" for a multigraph. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. For two vertices u, v of G we denote the number of distinct edges between u and v by $m_G(u,v)$ and we say that uv is an $m_G(u,v)$-edge. Degree of a vertex v of G is denoted by $\deg_G(v)$ (recall that in multigraphs, degree of a vertex v is defined as the number of edges going to other vertices plus twice the number of loops at v, i.e. $2m_G(v,v)+\sum_{u\neq v} m_G(u,v)$). By $N_G(v)$ we denote the multiset of neighbors of vertex v in G where the multiplicity of v in $N_G(v)$ is $2m_G(v,v)$ and for every $u \neq v$, the multiplicity of u is $m_G(u,v)$. We omit G in the subscript if G is clear from the context.

Suppose A and B are two multisets. Let A', resp. B' be the set of different elements from A, resp. B. We say that a mapping $g\colon A' \to B'$ is a bijection from A to B if for every $b' \in B'$ the sum of multiplicities of all elements from $g^{-1}(b')$ in A equals the multiplicity of b' in B. If C' is a set then by $A \cap C'$ we mean a multiset that contains only elements from $A' \cap C'$ with multiplicities corresponding to multiplicities in A. We denote the sum of multiplicities of all elements in A by $|A|$.

Let G and H be graphs. A *homomorphism* $f : V(G) \to V(H)$ is an edge preserving mapping from $V(G)$ to $V(H)$. A homomorphism f is a *covering projection* if $N_G(v)$ is mapped to $N_H(f(v))$ bijectively for every $v \in V(G)$. See Fig. 1 for an example. A covering projection is also known as a *locally bijective homomorphism* or simply a *cover*. In this paper we denote a covering projection f from G to H by $f\colon G \to H$.

Strictly speaking, as the notion of a covering projection stems from topology it should be defined by a pair of mappings – one on the vertices and one on the edges of the graphs involved. However, it was shown in [14] (using König's theorem and 2-factorization of $2k$-regular multigraphs) that every cover (as defined here) can be extended to a topological covering projection $f : V(G) \cup E(G) \to V(H) \cup E(H)$.

In this paper we consider the following decision problem.

Problem: H-Cover
Parameter: Fixed graph H.
Input: Graph G.
Task: Does there exist a covering projection $f : G \to H$?