Theoretical Computer Science 607 (2015) 337–350

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Optimally bracing grid frameworks with holes ☆

Yoshihiko Ito [a,*], Yuki Kobayashi [a], Yuya Higashikawa [a,1], Naoki Katoh [a,2], Sheung-Hung Poon [b,3], Maria Saumell [c,4]

[a] Department of Architecture and Architectural Engineering, Kyoto University, JST, CREST, Japan
[b] School of Computing and Informatics, Institut Teknologi Brunei, Gadong BE1410, Brunei Darussalam
[c] Department of Mathematics and European Centre of Excellence NTIS, University of West Bohemia, Czech Republic

ARTICLE INFO

Article history:
Received 8 April 2015
Received in revised form 23 July 2015
Accepted 31 July 2015
Available online 6 August 2015

Keywords:
Combinatorial rigidity
Bar-joint framework
Square grid framework
Bracing

ABSTRACT

We consider the bracing problem of a square grid framework possibly with holes and present an efficient algorithm for making the framework infinitesimally rigid by augmenting it with the minimum number of diagonal braces. This number of braces matches the lower bound given by Gáspár, Radics and Recski [2]. Our contribution extends the famous result on bracing the rectangular grid framework by Bolker and Crapo [1].

1. Introduction

Bolker and Crapo gave a necessary and sufficient condition in their seminal paper [1] for an $m \times n$ square grid framework with some diagonal braces of unit grid squares to be infinitesimally rigid. They defined a bipartite graph corresponding to the square grid framework with some diagonal braces such that the infinitesimal rigidity of the framework can be tested by checking the connectivity of the graph. In particular, the minimum number of diagonal braces that are necessary and sufficient to make the $m \times n$ square grid framework infinitesimally rigid is $m+n-1$ (see Fig. 1).

Radics and Recski [6] studied the case with holes where the outer boundary of the square grid framework is a simple rectilinear polygon, and long diagonal bars as well as cables can be used. They showed a lower bound for the number of diagonal bars and cables required to make the framework rigid; this bound matches the one for the case where only short braces (diagonal edges of unit grid squares) are allowed. However, in this case, they noted that the characterization based on a bipartite graph is no longer valid.

Gáspár, Radics and Recski [2] studied the case with holes where the outer boundary is rectangular, and derived a necessary and sufficient condition in terms of the rank of a certain matrix for an $m \times n$ square grid framework with some diagonal braces of unit grid squares to be infinitesimally rigid. The advantage of using the matrix introduced by [2] is that

☆ A preliminary version of this paper appeared in the proceedings of COCOA 2014 [4].
* Corresponding author.
E-mail address: as-ito-y@archi.kyoto-u.ac.jp (Y. Ito).
1 Supported by JSPS Grant-in-Aid for JSPS Fellows (26 · 4042).
2 Supported by JSPS Grant-in-Aid for Scientific Research(A) (25240004).
3 Supported in part by grant NSC 100-2628-E-007-020-MY3 in Taiwan, R.O.C.
4 Supported by the project LO1506 of the Czech Ministry of Education, Youth and Sports, and by the project NEXLIZ – CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

http://dx.doi.org/10.1016/j.tcs.2015.07.062
0304-3975/

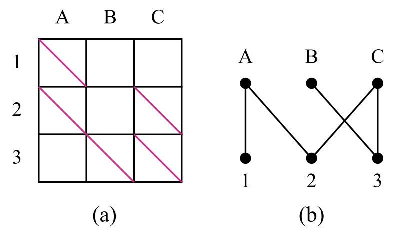

Fig. 1. A braced $m \times n$ square grid framework and the corresponding bipartite graph.

the matrix size is much reduced compared with using the original rigidity matrix, which helps to substantially reduce the running time for checking the rigidity. The paper [2] mentioned that the result can be generalized to the case where the outer boundary is a rectilinear simple polygon. However, the details were not given.

Although not clearly stated, the paper [6] mentioned that the above necessary and sufficient condition implies a lower bound on the number of diagonal bars and/or cables that are necessary to make the framework infinitesimally rigid, which is stated as

$$(\#\{\text{row and column segments}\}) - 2(\#\{\text{holes}\}) - 1 \tag{1}$$

where row and column segments will be defined later.

In this paper, we consider the case where short diagonal bars connecting opposite corners of a unit square (simply called *braces*) are used, and generalize the previous results in the following three directions.

1. We consider the case where there is no hole, but the outer boundary of the square grid polygon is a general rectilinear polygon. For this case, we shall give a characterization based on a bipartite graph which is the same as the one in [1].
2. For the case with holes, we observe that it is not possible to generalize the characterization in [1]. However, for this case we shall show a lower bound on the number of braces required to make the square grid framework infinitesimally rigid which matches the one obtained in [2].
3. We shall propose an algorithm for bracing a square grid framework using the minimum number of braces and matching the lower bound.

The motivation of the work by Bolker and Crapo [1] was finding a way to add braces to a one-story building to make it rigid. In view of this, it is a very important issue to investigate the bracing problem of grid frameworks with holes because walls and/or ceilings may have holes such as windows.

The rest of this paper is organized as follows. Section 2 introduces the necessary definitions and notations. Section 3 considers the case of a square grid framework with no holes such that the outer boundary is a general rectilinear polygon, and gives a characterization based on a bipartite graph which generalizes the result by [1]. Sections 4 and 5 consider the case of a square grid framework with holes such that the outer boundary is a general rectilinear polygon. In Section 4, we first show that the characterization using a bipartite graph is no longer possible, and then we give a lower bound on the number of braces required to make the framework rigid. Section 5 proposes an algorithm that adds the minimum number of braces required to make the framework rigid.

2. Preliminaries

We define a *grid framework* as a connected two-dimensional bar-joint framework which can be viewed as a union of unit grid squares. More formally, suppose we are given a rectilinear simple polygon P with holes $H_1, H_2, \ldots, H_h$ such that all vertices are located at the integer grid. We assume that (i) the area of every hole is at least two, (ii) the outer face of P and any hole H_j do not share any vertex, and (iii) any two distinct holes do not share any vertex. Let B_0 denote the outer boundary of P (i.e., the enclosing cycle of P) and let B_i (for $i = 1, 2, \ldots, h$) denote the boundary of H_i. Since all vertices are on integer grid points, the interior of P is decomposed into unit grid squares. We add all such unit grid squares (i.e., their vertices and edges) to the interior of P. The resulting framework is a grid framework which is denoted by F (see Fig. 2). We regard it as a bar-joint framework in which every edge is a rigid bar and every vertex is a universal joint (free joint) (see [7]). Let V and E denote the set of vertices and edges of F, respectively. Let S denote the set of squares of F. For any grid framework X, let V_X and E_X be the set of vertices and edges of X, respectively.

We define the set of *upper boundary squares*, denoted by S_u, as the set of squares s of S such that the upper edge of s is not shared with any other square of S (see Fig. 3(a)). Let $m = |S_u|$. Similarly, we define the set of *left boundary squares* S_l, and let $n = |S_l|$ (see Fig. 3(b)). For i with $0 \le i \le h$, let S_{ui} be the set of upper boundary squares whose upper edge is on the hole boundary B_i. Let $m_i = |S_{ui}|$ for $0 \le i \le h$. From our assumptions, the sets S_{ui} $(0 \le i \le h)$ are mutually disjoint, and thus $m = \sum_{i=0}^{h} m_i$ holds. For i with $0 \le i \le h$, let S_{li} be the set of left boundary squares whose left edges are on B_i. Let $n_i = |S_{li}|$ for $0 \le i \le h$. Notice that the sets S_{li} $(0 \le i \le h)$ are mutually disjoint, and thus $n = \sum_{i=0}^{h} n_i$ holds.