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Algorithms for fair partitioning of convex polygons

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ABSTRACT

In this paper we study the problem of partitioning a convex polygon P of n vertices into m polygons of equal area and perimeter. We give an algorithm for $m = 2$ that runs in $O(n)$ time, and an algorithm for $m = 2^k$, where k is an integer, that runs in $O((2n)^k)$ time. These are the first algorithms to solve this problem.

1. Introduction

In this paper we study the *Fair Partitioning of Convex Polygon* problem, which is stated as follows: Given a convex polygon P with n vertices, partition P into m disjoint polygons of equal area and perimeter.

The problem appears in many applications, including electrical engineering, convex optimizations, and special discretizations of spatial domains. The problem is related to the Cake Cutting Problem [1], where, given a convex polygon, one is supposed to find a partitioning in which all the sides have equal area (but perimeters can be different).

In electrical engineering, for instance, we are given a (convex) surface that is crossed by an electric current. We want to split the surface into a given number of smaller surfaces such that the contour of each of them has the same electric field, and the rate of change of the magnetic field in time over each of them is the same. By applying Maxwell's equations, this problem can be reduced to the Fair Partitioning of Convex Polygons problem.

An application of convex optimization would be the following: given a country with convex borders, divide the country into administrative regions (say, counties) and border the counties with fences. We want each county to have an equal share, so each one should receive the same area and the same border perimeter.

1.1. Our results

We present an optimal $O(n)$ time algorithm to find a fair partitioning of a polygon into $m = 2$ disjoint polygons. Note that, while a mathematical proof of existence of such partitioning was given in [4], this is the first algorithm to find such a partitioning. The algorithm is also optimal and we have implemented it. For $m = 2^k$, $k > 1$, we give an algorithm that runs in $O((2n)^k)$ time. Notice again that this is the first algorithm to find a fair partitioning for $m = 2^k$. We also have implemented this algorithm. Our algorithms are based on important observations and properties outlined in the following section.

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1.2. Related work

We start by giving some definitions and notions that we are going to use in the paper.

Definition 1. We call **left side** of a line L, the halfplane bounded by L that contains a point q that lies to the left of its projection on L, if L is not horizontal, and the halfplane above L, if L is horizontal. The other side of L is called **right side**.

Definition 2. Let L be a line that crosses polygon P, and $U(x, y)$ be the upper intersection point between L and P. We define the **left perimeter function** as follows: $LPF_{P,L}(x, y)$ is the perimeter of the left side of L inside P.

Definition 3. (See [4].) We call **area bisector** of a polygon P a line L that splits P into two sides of equal areas.

The cake-cutting problem, in which one is to find a partitioning of a circle into m parts of equal area, was studied by Steinhaus in 1948 [1]. This problem has extensions and relaxations. In one of them, there are m different measures μ_k of area on the polygon P, and each resulting sub-polygon P_k must have $\mu_k(P_k) >= 1/m$, given that $\mu_k(P) = 1$. In other words, if we are to split a convex polygonal cake to m people, everyone should believe they have received at least $1/m$ of it, according to their own measure [8]. In another version, the goal is to divide individual objects to m people, and we have the option of giving cash to compensate the final division. This was studied by Steinhaus in 1999 [8]. Bereg [11] proved the existence of a solution for another generalized version, namely alpha-partitioning, in which a convex shape (polygon or other) of area 1 is to be partitioned into 3 parts of areas $(a, a, 1 - 2a)$, where $0 < a < 1/3$. In 2003, Bereg also studied the 3-cutting of a convex shape with an arbitrary mass distribution [12], in which one is to partition the convex shape into 3 sectors that intersect in a point Q and have the same mass.

For the Optimal Convex Partitioning problem, we are given a simple polygon and we want to partition it into the smallest possible number of convex polygons, using only the initial polygon's vertices. This problem was first studied by Greene in 1983 [9]. The running time the author gives is $O(n^4)$. However, he also gives an approximation algorithm with an approximation ratio of 2, in $O(n \log n)$ time and $O(n)$ space. There is also an approximation algorithm by Hertel and Mehlhorn [10] that runs in $O(n)$ time and has an approximation ratio of 4.

The problem of fair partitioning of convex polygons was introduced in 2006 by Nandakumar and Rao [2,4]. The journal version of their paper was published in 2012 [3]. They conjectured that for any $m > 1$, a polygon can be convex fair partitioned (that is, all resulting m sides are convex polygons). They gave a proof for $m = 2$ and $m = 2^k$, which we are going to sketch. In 2010, Hubard and Aronov proved the conjecture for $m = p^k$, for any prime p and $k > 0$ [6]. Karasev studied the problem of Equipartition of Several Measures [7] where, given a convex shape and d measures in R^d, we want to find a partition into p^k convex parts that are equal in all measures, where p is a prime number. The Fair Partitioning problem is a special case of Equipartition of Several Measures where $d = 2$. In the following, we are going to briefly present the results in [2–6].

Theorem 1. *(See [3,4].) Given a convex polygon P, there is a convex fair partitioning of P into 2 sides of equal area and perimeter.*

To prove that, they first find an area bisector L. Such a line always exists due to continuity of the area of one side of L inside P. Then, they "rotate" L by changing its intersections with the polygon while keeping it an area bisector, until they get a fair partitioning. This eventually happens due to continuity of the left perimeter function $LPF_{P,L}$.

See Fig. 1 for an illustration.

Theorem 2. *(See [3,4].) Given a convex polygon P, there is a convex fair partitioning of P into 2^k sides of equal area and perimeter, for any $k > 1$.*

To prove that, they follow the same approach as for $m = 2$, except that, after finding an area bisector L, one has to recursively take each side of L and find a fair partitioning of it, then keep updating L and then recursively do the same operation on each side, until the resulting perimeters are equal.

Fig. 2 illustrates this proof for $m = 4$.

In 2009, Barany et. al. [5] gave a proof for $m = 3$, using a more complicated approach based on equivariant topologies. They use *convex 3-fans*, where a convex 3-fan is a set of 3 half-lines starting at a point q such that the sector in between any two of these half-lines is convex (its angle is less than π).

Theorem 3. *(See [5].) Given a convex polygon P, there is a convex fair partitioning of P into 3 sides of equal area and perimeter.*

See Fig. 3 for an illustration.

The page is organized as follows. We first give some important definitions, lemmas and observations concerning the problem, then we give our algorithmic solutions. In Section 3, we discuss the implementation of our algorithms, and in Section 4 we comment our results.