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Efficient approximation algorithms for bandwidth consecutive multicolorings of graphs

Yuji Obata, Takao Nishizeki *

Kwansei Gakuin University, 2-1 Gakuen, Sanda 669-1337, Japan

A R T I C L E I N F O

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A B S T R A C T

Let G be a graph in which each vertex v has a positive integer weight $b(v)$ and each edge (v, w) has a nonnegative integer weight $b(v, w)$. A bandwidth consecutive multicoloring, simply called a b-coloring of G, assigns each vertex v a specified number $b(v)$ of consecutive positive integers as colors of v so that, for each edge (v, w), all integers assigned to vertex v differ from all integers assigned to vertex w by more than $b(v, w)$. The maximum integer assigned to vertices is called the span of the coloring. The b-coloring problem asks to find a b-coloring of a given graph G with the minimum span. In the paper, we present four efficient approximation algorithms for the problem, which have theoretical performance guarantees for the computation time, the span of a found b-coloring and the approximation ratio. We also obtain several upper bounds on the minimum span, expressed in terms of the maximum b-degrees, one of which is an extension of Brooks' theorem on an ordinary coloring.

1. Introduction

An ordinary coloring of a graph G assigns each vertex a color so that, for each edge (v, w), the color assigned to v differs from the color assigned to w [19]. The problem of finding a coloring of a graph G with the minimum number $\chi(G)$ of colors often appears in the scheduling, task-allocation, etc. [9,17,19]; $\chi(G)$ is called the *chromatic number* of G. However, the problem is NP-hard, and it is difficult to find a good approximate solution. More precisely, for all $\epsilon > 0$, approximating the chromatic number $\chi(G)$ within $n^{1-\epsilon}$ is NP-hard [4,20], where n is the number of vertices in G.

In this paper we deal with a generalized coloring, called a "bandwidth consecutive multicoloring" [15]. Each vertex v of a graph G has a positive integer weight $b(v)$, while each edge (v, w) of G has a non-negative integer weight $b(v, w)$. A *bandwidth consecutive multicoloring* F of G is an assignment of positive integers to vertices such that

(a) each vertex v of G is assigned a set $F(v)$ of $b(v)$ consecutive positive integers; and
(b) for each edge (v, w) of G, all integers assigned to vertex v differ from all integers assigned to vertex w by more than $b(v, w)$, that is, $b(v, w) < |i - j|$ for any integers $i \in F(v)$ and $j \in F(w)$.

Most of the former problem formulations [5,11–14] except [15] require $b(v, w) \le |i - j|$ rather than $b(v, w) < |i - j|$, and hence our edge weight $b(v, w)$ is one less than the conventional one. A bandwidth consecutive multicoloring F is simply called a *b-coloring* for a weight function b. The maximum integer assigned to vertices is called the *span* of a b-coloring F,

* Corresponding author.
E-mail addresses: bji70496.7582@gmail.com (Y. Obata), nishizekit@gmail.com (T. Nishizeki).

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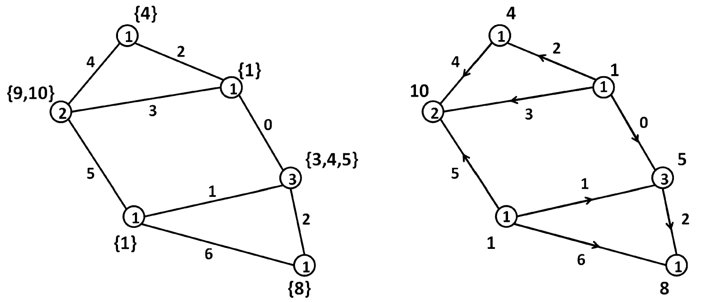

(a) (b)

Fig. 1. (a) A graph G and its optimal b-coloring F, and (b) the optimal b-coloring f corresponding to F and an acyclic orientation of G.

and is denoted by $\mathrm{span}(F)$. The *b-chromatic number* $\chi_b(G)$ of a graph G is the minimum span over all b-colorings of G. A b-coloring F is called *optimal* if $\mathrm{span}(F) = \chi_b(G)$. The *b-coloring problem* asks to find an optimal b-coloring of a given graph G. The ordinary vertex-coloring is merely a b-coloring such that $b(v) = 1$ for every vertex v and $b(v, w) = 0$ for every edge (v, w). The "bandwidth coloring" or "channel assignment" is a b-coloring such that $b(v) = 1$ for every vertex v [13–15]. The "multicoloring" is a b-coloring such that $b(v, w) = 0$ for every edge (v, w) and the set of integers assigned to a vertex are not necessarily consecutive. In Fig. 1(a), a vertex is drawn as a circle, in which the weight is written, while an edge is drawn as a straight line segment, to which the weight is attached. The b-chromatic number $\chi_b(G)$ of the graph G in Fig. 1(a) is 10, and an optimal b-coloring F of G with $\mathrm{span}(F) = 10$ is drawn in Fig. 1(a), where a set $F(v)$ is attached to each vertex v.

A b-coloring problem often arises in the assignment of radio channels of cellular communication systems [13–15] and in the non-preemptive task scheduling [17]. The $b(v)$ consecutive integers assigned to a vertex v correspond to the contiguous bandwidth of a channel v or a consecutive time period of a task v. The weight $b(v, w)$ assigned to edge (v, w) represents the requirement that the frequency band or time period of v must differ from that of w by more than $b(v, w)$. The span of an optimal b-coloring corresponds to the minimum total bandwidth or the minimum makespan.

The multicoloring problem can be solved in polynomial time for triangulated graphs and perfect graphs [1,7] and in pseudo polynomial time for graphs with bounded tree-width [8]. The bandwidth coloring problem and hence the b-coloring problem is NP-hard even for graphs with bounded tree-width [14,15], and hence there is no polynomial-time algorithm even for graphs with bounded tree-width unless P = NP. The b-coloring problem is strongly NP-hard for general graphs, and hence there is no FPTAS (fully polynomial-time approximation scheme) for general graphs. On the other hand, there is an FPTAS for graphs with bounded tree-width [15]. However, the computation time is very large; the FPTAS takes time $O(n^4/\varepsilon^3)$ even for series-parallel graphs, where n is the number of vertices in a graph and ϵ is the approximation error rate. For a bandwidth coloring or multicoloring, several heuristics using tabu search and genetic methodologies have been proposed and experimentally compared on their performances [5,11,12]. Thus, it is desirable to obtain an efficient approximation algorithm with theoretical performance guarantees for the b-coloring problem, which runs in linear time or $O(m \log n)$ time, where m is the number of edges in a graph. (In the paper we assume that all integers arithmetic can be done in constant time.)

In this paper we first present four efficient approximation algorithms for the b-coloring problem, which have theoretical performance guarantees for the computation time, the span of a found b-coloring and the approximation ratio. The first algorithm **Color-1** finds a b-coloring F with $\mathrm{span}(F) \le (c-1)\chi_b(G)$ in linear time when a given graph G is ordinarily vertex-colored with $c(\ge 2)$ colors. Hence, the approximation ratio of **Color-1** is at most $c - 1$, and is at most three particularly for planar graphs. The second algorithm **Color-2** is a variant of **Color-1**; **Color-2** tries all permutations of a given coloring of G. The third algorithm **Delta** finds a b-coloring F of a given graph G with $\mathrm{span}(F) \le \Delta_{1b}(G) + 1$ in time $O(m \log n)$, where $\Delta_{1b}(G)$ is a weighted version of the ordinary maximum degree $\Delta(G)$, called the "maximum uni-directional b-degree" of G. Thus $\chi_b(G) \le \Delta_{1b}(G) + 1$ for every graph G. The approximation ratio of **Delta** is at most $\Delta(G) + 1$. The fourth algorithm **Degenerate** finds a b-coloring F with $\mathrm{span}(F) \le k + 1$ in time $O(m \log \Delta(G))$ if G is a "(k, b)-degenerated graph", a weighted version of an ordinary k-degenerated graph [9]. It implies that $\chi_b(G) \le \Delta_{2b}(G) + 1$ for every graph G, where $\Delta_{2b}(G)$ is another weighted version of $\Delta(G)$, called the "maximum bi-directional b-degree" of G. The approximation ratio of **Degenerate** is at most $2\Delta(G) + 1$. We then show that an optimal b-coloring can be found in linear time for every graph G with $\Delta(G) \le 2$, and finally present a b-coloring analogue of the famous Brooks' theorem on an ordinary coloring [9,19]. An early version of the paper was presented at a workshop [16].

2. Preliminaries

In this section, we define several terms and present a known result.

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. Let $n = |V|$ and $m = |E|$ throughout the paper. For two integers α and β, we denote by $[\alpha, \beta]$ the set of all integers z with $\alpha \le z \le \beta$. Let $\mathbb{N}$ be the set of all positive integers, that are regarded as colors.