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Parameterized complexity of finding connected induced subgraphs

Leizhen Cai *,[1], Junjie Ye *

Department of Computer Science and Engineering, The Chinese University of Hong Kong, Shatin, Hong Kong SAR, China

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ABSTRACT

For a graph property Π, i.e., a family Π of graphs, the Connected Induced Π-Subgraph problem asks whether an input graph G contains k vertices V' such that the induced subgraph $G[V']$ is connected and satisfies property Π.

In this paper, we study the parameterized complexity of Connected Induced Π-Subgraph for decidable hereditary properties Π, and give a nearly complete characterization in terms of whether Π includes all complete graphs, all stars, and all paths. As a consequence, we obtain a complete characterization of the parameterized complexity of our problem when Π is the family of H-free graphs for a fixed graph H with $h \geq 3$ vertices: W[1]-hard if H is K_h, $\overline{K_h}$, or $K_{1,h-1}$; and FPT otherwise. Furthermore, we also settle the parameterized complexity of the problem for many well-known families Π of graphs: FPT for perfect graphs, chordal graphs, and interval graphs, but W[1]-hard for forests, bipartite graphs, planar graphs, line graphs, and degree-bounded graphs.

1. Introduction

Subgraph problems are central to graph algorithms and have been studied extensively with respect to both traditional and parameterized complexities [7,8]. For a graph property Π, i.e., a family Π of graphs, any graph in Π is a *Π-graph*, and the Induced Π-Subgraph problem asks whether the input graph contains an induced Π-subgraph with k vertices.

A classical result of Lewis and Yannakakis [10] states that Induced Π-Subgraph is NP-hard for any nontrivial hereditary property, and the problem remains NP-hard if we require the induced Π-subgraph to be connected. Khot and Raman [9] give a complete characterization of the parameterized complexity of Induced Π-Subgraph, with k being the parameter, depending on whether Π includes all complete graphs or trivial graphs (i.e., graphs without edges): W[1]-hard if Π includes all trivial graphs but not all complete graphs or vice versa, and FPT otherwise for decidable Π. In connection with this, Cai [2] showed earlier that the parametric dual of Induced Π-Subgraph (i.e., determining whether an n-vertex graph G contains an induced Π-graph on $n-k$ vertices, instead of k vertices) is FPT whenever Π can be characterized by a finite set of forbidden induced subgraphs.

In this paper, we investigate the parameterized complexity of the following induced Π-subgraph problems with the requirement that the k-vertex induced Π-graph is connected. Note that the work of Khot and Raman [9] does not address the issue of connectedness in induced Π-graphs. We will focus on hereditary properties Π.

* Corresponding authors.
E-mail addresses: lcai@cse.cuhk.edu.hk (L. Cai), jjye@cse.cuhk.edu.hk (J. Ye).
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Table 1
Parameterized complexity of Connected Induced Π-Subgraph for hereditary properties Π with sample properties, where FPT cases require Π to be also decidable.

Property Π	Include all complete graphs	Exclude some complete graphs
Include all stars	FPT (perfect graphs, chordal graphs, interval graphs)	W[1]-hard (forests, bipartite graphs, planar graphs)
Exclude some stars and include all paths	Unknown but W[1]-hard if Π includes all degree-bounded trees (claw-free graphs, line graphs)	W[1]-hard (degree-bounded graphs)
Exclude some stars and exclude some paths	W[1]-hard (co-planar graphs, co-bipartite graphs)	FPT (degree-bounded cographs)

Connected Induced Π-Subgraph
Instance: Graph G, positive integer k as parameter.
Question: Does G contain a connected induced Π-subgraph on k vertices?

The situation for Connected Induced Π-Subgraph is more complicated than that for Induced Π-Subgraph, and the parameterized complexity of the former largely depends on whether Π includes all complete graphs, stars, and paths (instead of complete graphs and trivial graphs for the latter). Table 1 summarizes our results for hereditary properties Π into six cases, with sample properties for each case.

For the remaining unknown case (Π includes all complete graphs and paths but excludes some stars), we show that it is W[1]-hard on bipartite graphs when paths are replaced by trees of maximum degree less than s, where $K_{1,s}$ is the smallest star excluded by Π. In order to show the intractability of this case, we prove that the classical Induced Path and Induced Cycle problems are both W[1]-hard on bipartite graphs.

Our results settle the parameterized complexity of Connected Induced Π-Subgraph for many well-known hereditary properties Π, including those listed in Table 1. Furthermore, our results also imply a complete characterization of the parameterized complexity of our problem when Π is the family of H-free graphs for a fixed graph H with $h \geq 3$ vertices: W[1]-hard if H is K_h, $\overline{K_h}$, or $K_{1,h-1}$; and FPT otherwise.[2]

All graphs in the paper are simple undirected graphs. We use K_t, $K_{1,s}$ and P_l, respectively, to denote the complete graph on t vertices, star on $1+s$ vertices, and path on l vertices. For a graph G, we use $V(G)$ to denote its vertex set and $E(G)$ its edge set. We use n and m, respectively, to denote the numbers of vertices and edges of G. For a subset $V' \subseteq V$, $N_G(V')$ denotes the neighbors of V' in $V(G) - V'$, and $G[V']$ represents the subgraph induced by V'. A *universal* vertex v of G is a vertex adjacent to all other vertices in G. A graph is *degree-bounded* if its vertex degree is bounded above by a constant. For a fixed graph H, a graph is H*-free* if it contains no induced subgraph isomorphic to H. For any Π-graphs, *co-*Π *graphs* denote complement graphs of Π-graphs. A property Π is *hereditary* if all induced subgraphs of a Π-graph are Π-graphs. It is well-known that Π is hereditary iff it can be characterized by a set of forbidden induced subgraphs, and we use $Forb(\Pi)$ to denote the minimum-size forbidden set of Π. Note that hereditary properties are not necessarily decidable.

In the paper, $R(t,s)$ denotes the Ramsey number, i.e., any graph with $R(t,s)$ vertices contains either a t-clique or an independent s-set. We use $M_{\Delta,D}$ to denote Moore's bound [13] — the maximum number of vertices in a connected graph G of maximum degree Δ and diameter D, i.e., $M_{\Delta,D} = 1 + \Delta \sum_{i=0}^{D-1} (\Delta - 1)^i < \Delta^{D+1}$ for $\Delta \geq 2$. We will use Moore's bound in the following way: if a connected graph G of maximum degree at most Δ contains Δ^{D+1} vertices, then G contains an induced path on $D+1$ vertices.

In the rest of the paper, we present FPT algorithms in Section 2, give W[1]-hardness proofs in Section 3, and consider the remaining case in Section 4. We discuss some open problems in Section 5.

2. FPT algorithms

We start with the two fixed-parameter tractable cases in Table 1. To obtain FPT algorithms for them, we use a combination of Ramsey's theorem, Moore's bound, and the random separation method of Cai, Chan and Chan [3].

Theorem 1. *Let Π be a decidable property. Then* Connected Induced Π-Subgraph *is FPT whenever*

1. *Π includes all complete graphs and stars, or*
2. *Π is hereditary and excludes some complete graphs, some stars, and some paths.*

Proof. By the assumption that Π is decidable, we may assume that it takes $T(k)$ time to determine whether a k-vertex graph is a Π-graph.

[2] Our conference paper [5] forgets to include disconnected H.