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Edge-clique covers of the tensor product

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ABSTRACT

In this paper we study the edge-clique cover number, $\theta_e(\cdot)$, of the tensor product $K_n \times K_n$. We derive an easy lowerbound for the edge-clique number of graphs in general. We prove that, when n is prime $\theta_e(K_n \times K_n)$ matches the lowerbound. Moreover, we prove that $\theta_e(K_n \times K_n)$ matches the lowerbound if and only if a projective plane of order n exists. We also show an easy upperbound for $\theta_e(K_n \times K_n)$ in general, and give its limiting value when the Riemann hypothesis is true. Finally, we generalize our work to study the edge-clique cover number of the higher-dimensional tensor product $K_n \times K_n \times \cdots \times K_n$.

1. Introduction

The edge-clique cover problem is the problem of determining if the set of edges of a graph can be expressed as a union of k cliques (i.e., if k cliques in the graph can cover all the edges in the graph). We denote by $\theta_e(G)$ the minimum number of cliques that are necessary to cover all its edges. For some graph classes like chordal graphs the edge-clique cover problem is polynomial time solvable [9,14]. However, finding a minimum edge-clique cover is NP-complete in some restricted graph classes like planar graphs [17].

It is known that the edge-clique cover problem is equivalent to finding a set representation of a graph G with at most k elements in the universe [5,19]. This number is also known as the intersection number [5,12]. For general graphs, the edge-clique cover problem does not have polynomial time approximation algorithms with factor less than 2 unless P = NP [13]. This paper concentrates on the edge-clique cover problem of the tensor product $K_n \times K_n$.

Unlike the clique cover problem, the edge-clique cover problem does not attract computer scientists' attention very much. However, the edge-clique cover problem is related to various applications in discrete mathematics, and more and more people started to conduct research on it [19]. For example, suppose that G is the *intersection graph* of a family of subsets of a set X. The minimal cardinality of X such that G is the intersection graph of a family of subset of X is equal to $\theta_e(G)$ [19].

The edge-clique cover problem is closely related to the *competition number* [11]. The competition number $k(G)$ of a graph G is the smallest number of isolated vertices which need to be added to G to make G into a competition graph. In ecology, we can use a competition graph to represent the competition between predators who prey on the same target. People started to ask what do the competition graphs of acyclic graphs look like. Roberts [18] found that by adding e isolated

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vertices to any graph G, where $e = |E(G)|$, the resulting graph becomes a competition graph of some acyclic graph. Further, Opsut [16] proved that $k(G)$ is bounded by

$$\theta_e(G) - n + 2 \le k(G) \le \theta_e(G),$$

where $n = |V(G)|$.

As mentioned, the research into edge-clique cover problem also made contributions to some other optimization problems. Kou et al. [13] showed that the optimization problem of keyword conflict is equivalent to the problem that finds the minimum number of cliques to cover the edges in a graph.

Finding the minimum edge-clique cover is hard. Gramm et al. [6] approached a method to reduce the number of vertices of G so that the edge-clique cover problem on G can be solved faster. Recently, Cygan et al. [3] proved that such an approach is optimal in running time by reducing 3-CNF-SAT formula with n variables and m clauses to an equivalent edge-clique cover instance (G, k) with $k = O(\log n)$ and $|V(G)| = O(n + m)$, provided that the Exponential Time Hypothesis holds. This paper focuses on the edge-clique cover problem of the tensor product of $K_n \times K_n$, which is one of the graph products that play an important role in graph decomposition into isomorphic subgraphs [8,10,15].

The rest of this paper is organized as follows. Section 2 introduces the tensor product $K_n \times K_n$. Section 3 derives a lowerbound for $\theta_e(K_n \times K_n)$. Sections 4 and 5, respectively, prove that $\theta_e(K_n \times K_n)$ matches the lowerbound when n is a prime and when a projective plane of order n exists. Section 6 derives an easy upperbound for $\theta_e(K_n \times K_n)$ in general, and gives its limiting value when the Riemann hypothesis is true. Section 7 discusses the edge-clique cover problem of the higher dimensional tensor product $K_n \times K_n \times \cdots \times K_n$.

2. The tensor product $K_n \times K_n$

Definition 1. Let G and H be two graphs. The *tensor product* $G \times H$ has

$$V(G) \times V(H) = \{(u, v) \mid u \text{ is a vertex in } G \text{ and } v \text{ is a vertex in } H\}$$

as its vertex set; two vertices (u, v) and (u', v') are adjacent in $G \times H$ if and only if

$$(u \text{ and } u' \text{ are adjacent in } G) \quad \text{and} \quad (v \text{ and } v' \text{ are adjacent in } H).$$

We write $[n] = \{1, \dots, n\}$. We refer to 'gridpoints' as the elements of the set

$$V = \{\, (i, j) \mid i \in [n] \text{ and } j \in [n] \,\}.$$

We also write the set V as $[n] \times [n]$.

Corollary 1. *The tensor product $K_n \times K_n$ has V as its set of vertices. Two vertices (i, j) and (k, ℓ) are adjacent if and only if*

$$i \ne k \quad \text{and} \quad j \ne \ell.$$

Notice that the tensor product is the complement of the Cartesian product $K_n \Box K_n$ also known as a rook's graph. In this product two pairs (i, j) and (k, ℓ) are adjacent if they lie in the same 'row' or 'column,' that is, the pairs are adjacent if either

$$(i = k \quad \text{and} \quad j \ne \ell) \quad \text{or} \quad (i \ne k \quad \text{and} \quad j = \ell).$$

3. Edge-clique covers

We are interested in the edge-clique cover problem (especially in graph limits). In this paper we concentrate on the edge-clique cover problem of the tensor product $K_n \times K_n$.

Definition 2. Let G be a graph. An *edge-clique cover* $\mathcal{C}$ is a set of cliques such that each edge of G has both its endpoints in at least one element of $\mathcal{C}$.

We denote by $\theta_e(G)$ the minimal number of cliques in an edge-clique cover of G. This number is also known as the intersection number.

Definition 3. Let G be a graph. An *edge-clique partition* is a collection of cliques $\mathcal{C}$ such that each edge is in precisely one element of $\mathcal{C}$.

We denote by $ecp(G)$ the minimal number of cliques in an edge-clique partition of G.