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Computing the Tutte polynomial of lattice path matroids using determinantal circuits

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ABSTRACT

We give a quantum-inspired $O(n^4)$ algorithm computing the Tutte polynomial of a lattice path matroid, where n is the size of the ground set of the matroid. Furthermore, this can be improved to $O(n^2)$ arithmetic operations if we evaluate the Tutte polynomial on a given input, fixing the values of the variables. The best existing algorithm, found in 2004, was $O(n^5)$, and the problem has only been known to be polynomial time since 2003. Conceptually, our algorithm embeds the computation in a determinant using a recently demonstrated equivalence of categories useful for counting problems such as those that appear in simulating quantum systems.

1. Introduction

Since their introduction in the early 20th century, matroids have proven to be immensely useful objects generalizing notions of linear independence. They have become ubiquitous, appearing in fields from computer science and combinatorics to geometry and topology (cf. [9,29]).

Perhaps the most famous invariant of a matroid M is called the Tutte polynomial, $T_M(x, y)$. The polynomial was originally defined as an invariant of graphs, generalizing the chromatic polynomial [36]. This was later discovered to specialize to the Jones polynomial of an associated alternating knot ([35]) as well as the partition function of the Potts model in statistical physics, the random cluster model in statistical mechanics ([15]), the reliability polynomial in network theory ([13]), and flow polynomials in combinatorics ([41]). In fact, it is the most general invariant of matroids such that $F(M \oplus M') = F(M)F(M')$; all other such invariants are evaluations of the Tutte polynomial [42].

The Tutte polynomial specializes to a generating function of special configurations of chip firing games on graphs [25]. In convex geometry, it also relates to the Ehrhart polynomial of zonotopes which is used for calculating integral points in polytopes [26]. More recently the connection with quantum simulation and computation has been explored, but without obtaining a new classical algorithm for the Tutte polynomial of lattice path matroids [33,43,1,2].

In addition to generalizing several polynomials, specific values of the Tutte polynomial give information about graphs, including the number of (spanning) forests, spanning subgraphs, and acyclic orientations. However, computing the Tutte polynomial for general matroids is very difficult. For x, y positive integers, calculating $T_M(x, y)$ is #P-hard [19]. As such, a large amount of work has been done to determine when Tutte polynomials can be efficiently computed.

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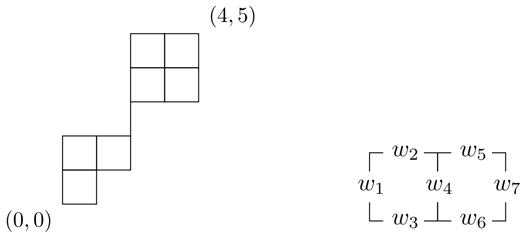

(a) Lattice bounded by two monotone paths P (lower) and Q (upper).

(b) Weighted lattice.

Fig. 1. Lattices.

Lattice path matroids were presented in [7,5] as a particularly well-behaved and yet very interesting class of matroids. For lattice path matroids, the computation of the Tutte polynomial was shown to be polynomial time in the 2003 paper [5]. In [6], it was proven that the time complexity of computing the Tutte polynomial is $O(n^5)$, where n is the size of the ground set of the matroid.

We give a quantum-inspired $O(n^4)$ algorithm computing the Tutte polynomial of a lattice path matroid, where n is the size of the ground set of the matroid. Furthermore, this can be improved to $O(n^2)$ arithmetic operations (as opposed to bit operations) if we evaluate the Tutte polynomial on a given input, fixing the values of the variables. The best existing algorithm was $O(n^5)$. Our algorithm embeds the computation in a determinant using a recently demonstrated equivalence of categories [28] useful for counting problems such as those addressed by holographic algorithms.

2. Background

Leslie Valiant defined matchgates to simulate certain quantum circuits efficiently [39]. He then defined holograph algorithms as a way of finding polynomial-time algorithms for certain #P problems [37]. He later went on to demonstrate several new polynomial-time algorithms for problems for which no such algorithms were previously known [40]. These algorithms came to be known as matchcircuits and have been studied extensively [11,38,8,22]. They were reformulated later in terms of tensor networks and are equivalent to Pfaffian circuits [23,27].

Tensor networks were probably first introduced by Roger Penrose in the context of quantum physics [30], but appear informally as early as Cayley. They have seen many applications in physics, as they can represent channels, maps, states and processes appearing in quantum theory [16–18,4].

Tensor networks also generalize notions of circuits, including quantum circuits. As such, they have seen an increase in popularity as tools for studying complexity theory [10,14,12]. While we do not define tensor networks in this paper, the material should be understandable without a detailed knowledge of this formalism. For more detailed expositions on tensor networks, see [32,21,20,3].

2.1. Structure

In [28], a new type of circuit was defined based on determinants, as opposed to matchcircuits which are defined in terms of Pfaffians. It was shown that these circuits had polynomial time evaluations. In this paper, we give an algorithm that improves the complexity of computing the Tutte polynomial of a lattice path matroid to $O(n^4)$ using these *determinantal circuits*. Then we show that evaluating the Tutte polynomial on a specific input (fixed values of x and y) can be done in $O(n^2)$ arithmetic operations. The paper is organized as follows: first we discuss weighted lattice paths and their relation to determinantal circuits. Then we recall the definitions of lattice path matroids and the relevant theorems from [5,6]. Lastly, we give an explicit algorithm for computing the Tutte polynomial and analyze its time complexity.

3. Weighted lattice paths

Let us consider $\mathbb{Z}^2$ as an infinite graph where two points are connected if they differ by $(\pm 1, 0)$ or $(0, \pm 1)$. Suppose we are given two monotone paths on $\mathbb{Z}^2$, P and Q, that both start at $(0, 0)$ and end at (m, r). Furthermore, suppose that P is never above Q in the sense that there are no points $(p_1, p_2) \in P$, $(q_1, q_2) \in Q$ such that $p_1 - q_1 < 0$ and $p_2 - q_2 > 0$. We are interested in subgraphs of $\mathbb{Z}^2$ bounded by such pairs of paths. From here on out, "lattice" means of subgraph of this form. An example is given by Fig. 1(a).

Let E be the set of edges of a lattice G. Suppose for each $e \in E$, we assign it a weight, $w(e)$. We call this a *weighted lattice*. Given a monotone path $C \subseteq G$, we define the weight of C to be the product of its edge weights

$$w(C) = \prod_{e \in C} w(e).$$