Theoretical Computer Science 597 (2015) 1–17

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

The decidability of the intensional fragment of classical linear logic

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ARTICLE INFO

Article history:
Received 14 July 2014
Received in revised form 23 March 2015
Accepted 5 June 2015
Available online 16 June 2015
Communicated by A. Avron

Keywords:
Cognate sequent
Curry's lemma
Decidability
König's lemma
Kripke's lemma
Linear logic
Modal logic
Relevance logic
Sequent calculus

ABSTRACT

The intensional fragment of classical propositional linear logic combines modalities with contraction-free relevance logic — adding modalized versions of the thinning and contraction rules. This paper provides a *proof of the decidability* of this logic based on a sequent calculus formulation. Some related logics and some other fragments of linear logic are also shown decidable.

1. Introduction

Linear logic was introduced by Girard [15], and it has been applied, reformulated, extended and thoroughly investigated.[1] Theorem 3.7 in Lincoln et al. [24] claims that the propositional fragment is undecidable. However, the undecidability of linear logic would not imply the undecidability of its fragments. Indeed, it seems that the decidability of the intensional fragment (sometimes termed as *MELL*) has not been solved.[2]

Kripke [20] has shown that certain *relevance logics* are decidable. Detailed presentations of those results may be found in Belnap and Wallace [6], Anderson and Belnap [2, §13] and Dunn [13, §§3.6–3.9]. Meyer [27] proved that the non-distributive logic of relevant implication is decidable. The result has been further elaborated on in Thistlewaite et al. [35]. Bimbó and Dunn [10] extended the scope of decidable fragments to include the implicational logic of ticket entailment, and Bimbó [8, Ch. 9] provides proofs of decidability for further logics. This paper shows — using a version of the method exemplified by these results — that the *intensional fragment* of classical propositional linear logic is decidable. The modalization of the structural rules is peculiar to linear logic, which adds an extra layer to the decidability proof.

In Section 2, we carefully formulate sequent calculi for CLL_{int}, the intensional fragment of classical linear logic and RLL_{int}, intensional interlinear logic. The former is at the center of this paper, however, the latter is used in an essential way in

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[1] See also Belnap [5], Girard [16] and [17], Gunter and Gehlot [18], Kopylov [19], Lafont [22], Lincoln and Winkler [25], Martini and Masini [26], Meyer et al. [31], Nigam and Miller [32], Urquhart [37] and [38].

[2] *MELL* is a label used, for example, in Lincoln [23] and Di Cosmo and Miller [12]. The decidability of *MELL* is listed as an open problem by Y. Lafont on his web pages at the URL iml.univ-mrs.fr/~lafont/linear/decision/bienvenue.html (accessed on March 15th, 2015).

http://dx.doi.org/10.1016/j.tcs.2015.06.019
0304-3975/

the decidability proof of CLL_{int}. Section 3 provides proofs — in some detail — of the cut theorem for these calculi. These theorems ensure that the sequent calculi are properly formulated, and that cut-free proofs are sufficient. Section 4 contains the proof of the decidability of CLL_{int}, with an auxiliary proof of the decidability of RLL_{int}. In Section 5, we prove similar results for logics closely related to CLL_{int} and RLL_{int}. In the last section, we provide some concluding remarks about the importance of our results.

2. Sequent calculi

We introduce four sequent calculi that formalize two logics, the intensional fragment of linear logic and intensional interlinear logic.

2.1. Intensional fragment of classical linear logic: CLL_{int}

The intensional fragment of classical propositional linear logic is denoted by CLL_{int}.[3] The language of this logic contains a unary connective $^{\perp}$ (*negation*), three binary connectives $\multimap$ (*implication*), $\otimes$ (*fusion*) and $\parr$ (*fission*), and two unary modalities ! (*possibility*) and ? (*necessity*).[4] The atomic formulas comprise a denumerable set of *propositional variables*; *formulas* are defined by the following CFG, with the proviso that $\mathbb{P}$ is a non-terminal symbol that rewrites to a propositional variable.

$$\mathcal{A} ::= \mathbb{P} \mid (\mathcal{A}^{\perp}) \mid (\mathcal{A} \multimap \mathcal{A}) \mid (\mathcal{A} \otimes \mathcal{A}) \mid (\mathcal{A} \parr \mathcal{A}) \mid !\mathcal{A} \mid ?\mathcal{A}$$

We use $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ as *metavariables* for formulas. Capital Greek letters stand for *multisets of formulas* (including the empty multiset of formulas).[5] The multiset $[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ is denoted by $\mathcal{A}_1; \ldots; \mathcal{A}_n$; that is, we omit the brackets and separate the elements of the multiset by semicolons. This notation is in harmony with standard notation in sequent calculi for non-classical logics. $\Gamma; \Delta$ is a shorthand for the union of Γ and Δ, and $\mathcal{A}; \Gamma$ (or $\Gamma; \mathcal{A}$) is the union of Γ and $[\mathcal{A}]$ (the singleton multiset containing one copy of $\mathcal{A}$). The superscript modalities on multisets in the rules below indicate that in order for a rule to be applicable the formulas in the multiset must be appropriately modalized. For instance, $\Gamma^!$ is a multiset of formulas, in which each element $\mathcal{C}$ is of the form $!\mathcal{A}$, for some $\mathcal{A}$.

A *sequent* is a pair of multisets of formulas separated by $\vdash$. The *sequent calculus* CLL_{int} comprises the following *axiom* and *rules*. The last four rules are modalized structural rules. The rest of the rules are connective rules.

$$\mathcal{A} \vdash \mathcal{A} \ \text{id}$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A}}{\mathcal{A}^{\perp}; \Gamma \vdash \Delta}\ {}^{\perp}\vdash \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta}{\Gamma \vdash \Delta; \mathcal{A}^{\perp}}\ \vdash^{\perp}$$

$$\frac{\mathcal{A}; \mathcal{B}; \Gamma \vdash \Delta}{\mathcal{A} \otimes \mathcal{B}; \Gamma \vdash \Delta}\ \otimes\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \Theta \vdash \Lambda; \mathcal{B}}{\Gamma; \Theta \vdash \Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}}\ \vdash\otimes$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda}\ \multimap\vdash \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \multimap \mathcal{B}}\ \vdash\multimap$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda}\ \parr\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \parr \mathcal{B}}\ \vdash\parr$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !\vdash \qquad \frac{\Gamma^! \vdash \Delta^?; \mathcal{A}}{\Gamma^! \vdash \Delta^?; !\mathcal{A}}\ \vdash!$$

$$\frac{\mathcal{A}; \Gamma^! \vdash \Delta^?}{?\mathcal{A}; \Gamma^! \vdash \Delta^?}\ ?\vdash \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?$$

$$\frac{\Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !K\vdash \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?K$$

$$\frac{!\mathcal{A}; !\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\ !W\vdash \qquad \frac{\Gamma \vdash \Delta; ?\mathcal{A}; ?\mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}\ \vdash?W$$

The notion of a *proof* is usual; that is, a proof is a tree of occurrences of sequents where the leaves are instances of the axiom, and parent nodes are obtained by an application of a rule to their children. The term "tree" in this paper means a

[3] "Intensional" is used in accordance with the use of this term in Bimbó and Dunn [9]. CLL_{int} is or is very closely related to *MELL* — depending on what exactly is meant by that acronym.

[4] As a compromise between the unconventional notation of Girard [15] and the standard terminology in non-classical logics, I use Girard's symbols with the customary names of the connectives. A translation may be found in Avron [3], though the modalities here are switched for semantic reasons as explained in Bimbó and Dunn [9, Ch. 3].

[5] By "multiset" we always mean a *finite* multiset; hence, we drop "finite." Many logics can be formulated as sequent calculi based on multisets — see e.g., Meyer and McRobbie [29] and [30] for related logics.