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Improved approximation algorithms for a bilevel knapsack problem

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ABSTRACT

We study the Stackelberg/bilevel knapsack problem as proposed by Chen and Zhang [1]: Consider two agents, a leader and a follower. Each has his own knapsack. (Knapsack capacities are possibly different.) As usual, there is a set of items $i = 1, \dots, n$ of given weights w_i and profits p_i. It is allowed to pack item i into both knapsacks, but in this case the corresponding profit for each player becomes $p_i + a_i$, where a_i is a given (positive or negative) number. The objective is to find a packing for the leader such that the total profit of the two knapsacks is maximized, assuming that the follower acts selfishly. We present tight approximation algorithms for all settings considered in [1].

1. Introduction

The standard knapsack problem is one of the most fundamental and well-studied problems in combinatorial optimization: There is a knapsack of prescribed capacity W and n items with given size w_i and profit p_i. The task is to select a set of items of total size at most W and maximum total profit. A first bilevel variant (in the form of a *Stackelberg game*) was introduced by Dempe and Richter [2]: There are two decision makers (players) – a *leader* and a *follower* – as well as a (universal) knapsack with flexible capacity and a set of items with given sizes as above, yet item profits may vary w.r.t. the leader and the follower, respectively. The leader first determines the capacity of the knapsack, and afterwards the follower, assumed to be selfish, packs items to the knapsack, maximizing his own profit. The (leader's bilevel) problem is to compute the knapsack capacity such that the leader's profit – defined by a linear function of the knapsack capacity plus his total profit of packed items – is maximized.

Several other bilevel variants of knapsack have been proposed as well. For example, Mansi et al. [12] study a setting in which both the leader and the follower pack items into a knapsack (of fixed capacity). DeNegre [17] investigates a bilevel version where both players own a private knapsack each and pack items from a common item set. Again, the leader acts first, selecting a set of items for his own knapsack, then the follower packs items from the remaining item set into his own knapsack, seeking to maximize his total profit. The objective of the (hostile) leader is to choose his set of items such that the follower's profit is minimized.

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Cases		Approx. ratios [1]	Lower bounds
$a_i \le 0$		$2+\epsilon$	1.5
$a_i \ge 0$	$W_1 > W_2$	$1+\sqrt{2}+\epsilon$	1.5
	$W_1 < W_2$	$2+\epsilon$	2

Fig. 1. Known lower bounds.

In this paper we consider yet another variant of the bilevel knapsack problem, due to Chen and Zhang [1]. In this setting, again, each player has his own knapsack of fixed capacities W_1 and W_2, respectively. Items $1, \dots, n$ have fixed weights w_i and profits p_i. The characteristic feature of the model in [1] is that items may be *double-packed*, *i.e.* packed by both players. In case item i is packed only by one player, it accounts for a profit of p_i, as usual, however, if i is packed by both players, its profit (for both players) is modified to $p_i + a_i$ for given *profit modifier* $a_i \in \mathbb{R}$. Again, the setting is that of a Stackelberg game, and the objective is to exhibit an optimal packing for the leader, *i.e.*, one that maximizes the total profit assuming that the second player (the follower) acts selfishly (disregarding the impact any double packing may have on the items packed by the leader). As a motivating example, Chen and Zhang mention the case of two investors, say, the government and a company with budgets W_1 and W_2, respectively. Items correspond to potential projects of cost w_i and reward p_i, resp. $p_i + a_i$ with $a_i > 0$ if both players invest in project i. Depending on the application, the numbers a_i may be positive or negative ("double booking"). In case all a_i are positive, Chen and Zhang [1] call it the *beneficial model* and if all a_i are negative, it is referred to as the *competitive model*.

Bilevel optimization is often computationally difficult and likely to extend beyond NP. In the last decades, bilevel and multilevel optimization have received much attention in the literature (*cf.* books by Migdalas, Pardalos and Värbrand [4] and Dempe [3], a survey by Colson et al. [5]). Dempe and Richter [2] introduced a mixed integer bilevel program for their problem variant and proposed an algorithm based on branch and bound. Afterwards, a dynamic programming algorithm for this problem was given by Brotcorne et al. [6]. Recently, Caprara et al. [7] proved that the first three problem variants mentioned above are Σ_2^P-hard (probably the fourth one is as well), *i.e.*, there is no way of formulating them as single-level integer programs of polynomial size unless *the polynomial hierarchy collapses* (*cf.* [7] for more details). In particular, they showed that the first two variants (*cf.* Dempe and Richter [2], Mansi et al. [12]) do not possess a polynomial approximation algorithm with finite worst case guarantee unless $P = NP$ and proposed a polynomial time approximation scheme for the third variant (*cf.* DeNegre [17]), which is known as the first approximation scheme for a Σ_2^P-hard problem. For other variants and related problems, *cf.* [8–12].

Regarding the problem to be considered in this paper, Chen and Zhang [1] proposed a $(2+\epsilon)$-approximation algorithm for the competitive model ($a_i \le 0$), and, for the beneficial model ($a_i \ge 0$), a $(1+\sqrt{2}+\epsilon)$-approximation for the case $W_1 > W_2$ and a $(2+\epsilon)$-approximation for the case $W_1 \le W_2$.

In this paper, we present better approximation algorithms for the beneficial model as well as the competitive model and show that the approximation ratios are tight in each case, *i.e.*, the approximation ratios can be made arbitrarily close to the known lower bounds (*cf.* Fig. 1). The main ingredients of our approach are: An ϵ-approximation of the maximum profit problem in case both players cooperate – which may be of independent interest, *cf.* (P3) in Section 2 – and a factor revealing LP for estimating the quality of our approximation algorithms (*cf.* Jain et al. [18]).

The rest of the paper is organized as follows: In the section below, we formally introduce the bilevel problem (*cf.* (P1) in Section 2) and its "cooperative" counterpart (*cf.* (P3)). In Section 3, we describe a polynomial time approximation scheme (PTAS) for the cooperative problem version (P3). In Section 4, we present new approximation algorithms and analyze their approximation ratios. Finally, in Section 5, we mention some open problems.

2. Bilevel knapsack with independent knapsacks

Let W_1, W_2 be capacities of the knapsacks owned by player 1 (leader) and player 2 (follower), respectively. Let $A = \{1, 2, \dots, n\}$ be a set of items of weight w_i, profit p_i and "double packing modifier" a_i for all $i \in A$. Let $x_i, y_i \in \{0, 1\}$ indicate whether item i is packed by player 1 and player 2, respectively.

Recall that the profit of item i is modified to $p_i + a_i$ if i is packed by both players. Thus the leader's problem can be formulated as a bilevel integer program as follows:

$$\max_x \sum_{i=1}^n p_i(x_i + y_i) + 2\sum_{i=1}^n a_i x_i y_i \tag{P1}$$

$$\text{s.t.} \sum_{i=1}^n w_i x_i \le W_1,$$

$$x_i \in \{0,1\}, \quad i = 1, 2, \dots, n,$$

y is an optimal solution of (P2) *below*.